sukita1729@gmail.com

§ 6.6 Inverse Trigonometric Functions

＊ Inverse trigonometric functions - Definition & Derivatives

Since $y = \sin x$ is not a one-to-one function, we restrict the domain of the function on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then, $y = \sin x$ becomes a one-to-one function and we can define its inverse function.

◎ Def> $y = \sin^{-1}x$ or $y = \arcsin x \Leftrightarrow x = \sin y$,

$dom f_1 = \{x \mid -1 \le x \le 1\}$, $ran f_1 = \left\{y \,\middle|\, -\frac{\pi}{2} \le y \le \frac{\pi}{2}\right\}$

$y = \cos^{-1}x$ or $y = \arccos x \Leftrightarrow x = \cos y$,

$dom f_2 = \{x \mid -1 \le x \le 1\}$, $ran f_2 = \{y \mid 0 \le y \le \pi\}$

$y = \tan^{-1}x$ or $y = \arctan x \Leftrightarrow x = \tan y$,

$dom f_3 = \{x \mid x \in \mathbb{R}\}$, $ran f_3 = \left\{y \,\middle|\, -\frac{\pi}{2} < y < \frac{\pi}{2}\right\}$

ex) $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$, $\tan\left(\arcsin\frac{1}{3}\right) = \frac{1}{2\sqrt{2}}$

◎ Thm> $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

pf) $y = \sin^{-1}x \Leftrightarrow x = \sin y$, $1 = \cos y \cdot y'$, $y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$ ■

$y = \cos^{-1}x \Leftrightarrow x = \cos y$, $1 = -\sin y \cdot y'$, $y' = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-x^2}}$ ■

$y = \tan^{-1}x \Leftrightarrow x = \tan y$, $1 = \sec^2 y \cdot y'$, $y' = \frac{1}{\sec^2 y} = \frac{1}{1+x^2}$ ■

ex) $f(x) = \sin^{-1}(x^2-1)$, $dom f = \{x \mid -\sqrt{2} \le x \le \sqrt{2}\}$,

$f'(x) = \frac{2x}{\sqrt{-x^4+2x^2}} = \frac{2x}{|x|\sqrt{2-x^2}}$, $dom f' = \{x \mid -\sqrt{2} < x < \sqrt{2},\, x \ne 0\}$

◎ Def> $y = \csc^{-1}x$ or $y = \mathrm{arc\,csc}\,x \Leftrightarrow x = \csc y$,

$dom f_4 = \{x \mid |x| \ge 1\}$, $ran f_4 = \left\{y \,\middle|\, 0 < y \le \frac{\pi}{2} \vee \pi < y \le \frac{3}{2}\pi\right\}$

$y = \sec^{-1}x$ or $y = \mathrm{arc\,sec}\,x \Leftrightarrow x = \sec y$,

$dom f_5 = \{x \mid |x| \ge 1\}$, $ran f_5 = \left\{y \,\middle|\, 0 \le y < \frac{\pi}{2} \vee \pi \le y < \frac{3}{2}\pi\right\}$

$y = \cot^{-1}x$ or $y = \mathrm{arc\,cot}\,x \Leftrightarrow x = \cot y$,

$dom f_6 = \{x \mid x \in \mathbb{R}\}$, $ran f_6 = \{y \mid 0 < y < \pi\}$

◎ Thm> $\frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$

pf) $y = \csc^{-1}x \Leftrightarrow x = \csc y$, $1 = -\csc y \cot y \cdot y'$,

$y' = -\frac{1}{\csc y \cot y} = -\frac{1}{x\sqrt{x^2-1}}$ ■

$y = \sec^{-1}x \Leftrightarrow x = \sec y$, $1 = \sec y \tan y \cdot y'$,

$y' = \frac{1}{\sec y \tan y} = \frac{1}{x\sqrt{x^2-1}}$ ■

$y = \cot^{-1}x \Leftrightarrow x = \cot y$, $1 = -\csc^2 y \cdot y'$,

$y' = -\frac{1}{\csc^2 y} = -\frac{1}{1+x^2}$ ■

ex) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

pf) Let $f(x) = \tan^{-1}x + \cot^{-1}x$, $f'(x) = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$ and $f(x) = c \in \mathbb{R}$.

If $x = 1$ then $f(1) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$. $\therefore f(x) = \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ ■

＊ Inverse trigonometric functions – Antiderivatives

◎ Thm> $\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}x + C$

$\int \frac{1}{1+x^2}dx = \tan^{-1}x + C$

ex) $\int_0^{\frac{1}{4}} \frac{1}{\sqrt{1-4x^2}}dx = \int_0^{\frac{1}{2}} \frac{1}{2\sqrt{1-u^2}}du = \left[\frac{1}{2}\sin^{-1}u\right]_0^{\frac{1}{2}} = \frac{\pi}{12}$ ■ $(u = 2x,\ du = 2dx)$

ex) $\int \frac{1}{x^2+a^2}dx = \frac{1}{a^2}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$ ■

§ 6.7 Hyperbolic Functions

＊ Hyperbolic functions – Definition & Derivatives

◎ Def> $\sinh x = \frac{e^x - e^{-x}}{2}$ $(y \in \mathbb{R})$

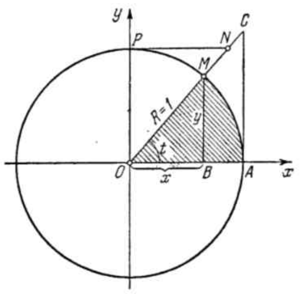
$\cosh x = \frac{e^x + e^{-x}}{2}$ $(y \geq 1)$

$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ $(-1 < y < 1)$

$\operatorname{csch} x = \frac{1}{\sinh x}$ $(y \neq 0)$

$\operatorname{sech} x = \frac{1}{\cosh x}$ $(0 < y \leq 1)$

$\coth x = \frac{1}{\tanh x}$ $(|y| > 1)$

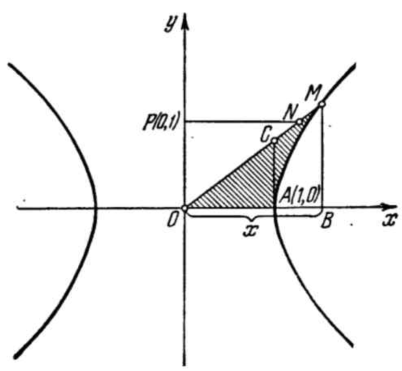


$x^2 + y^2 = 1$

x = cos t , y = sin t , t = ∠MOA

Area OMA = 1/2 × arc $\overset{\frown}{MA}$ × R.
Arc $\overset{\frown}{MA}$ = R × t
So Area OMA = 1/2 × R × R × t = (1/2) × t
because R = 1.
Hence, **t** = 2 × Area OMA,

Denoting **area OMA = A (hatched area),**
we have **t = 2A**



$x^2 - y^2 = 1$

x = cosh t, y = sinh t

t = 2 × area OMA =
2 × (area OMB – area AMB)
Denoting **area OMA = A (hatched area),**
we have **t = 2A**

sukita1729@gmail.com

◎ Thm> Identities of hyperbolic functions

$\sinh(-x) = -\sinh x$	$\sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y$
$\cosh(-x) = \cosh x$	$\cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y$
$\cosh^2 x - \sinh^2 x = 1$	$\sinh(2x) = 2\sinh x\cosh x$
$\tanh^2 x = 1 - \operatorname{sech}^2 x$	$\cosh(2x) = \cosh^2 x + \sinh^2 x$
$\coth^2 x = 1 + \operatorname{csch}^2 x$	$\cosh^2 x = \dfrac{\cosh(2x)+1}{2}$
	$\sinh^2 x = \dfrac{\cosh(2x)-1}{2}$

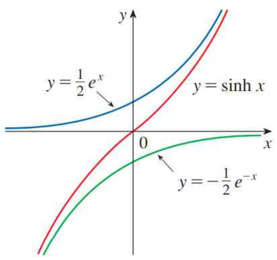


FIGURE 1
$y = \sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

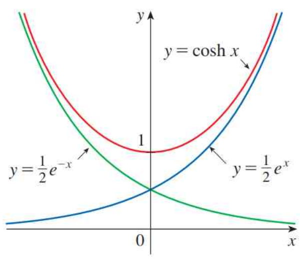


FIGURE 2
$y = \cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

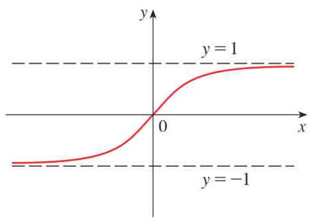


FIGURE 3
$y = \tanh x$

◎ Thm>

$\dfrac{d}{dx}(\sinh x) = \dfrac{e^x + e^{-x}}{2} = \cosh x$	$\dfrac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x\coth x$
$\dfrac{d}{dx}(\cosh x) = \dfrac{e^x - e^{-x}}{2} = \sinh x$	$\dfrac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x\tanh x$
$\dfrac{d}{dx}(\tanh x) = \dfrac{2^2}{(e^x + e^{-x})^2} = \operatorname{sech}^2 x$	$\dfrac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

＊ Inverse hyperbolic functions – Definition & Derivatives

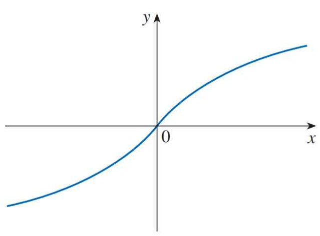


FIGURE 8 $y = \sinh^{-1} x$
domain = $\mathbb{R}$ range = $\mathbb{R}$

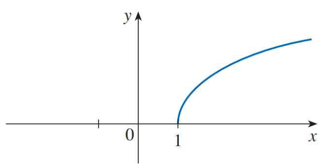


FIGURE 9 $y = \cosh^{-1} x$
domain = $[1, \infty)$ range = $[0, \infty)$

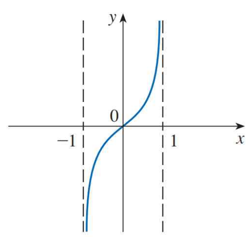


FIGURE 10 $y = \tanh^{-1} x$
domain = $(-1, 1)$ range = $\mathbb{R}$

◎ Thm> $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$, $\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$, $\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$

sukita1729@gmail.com

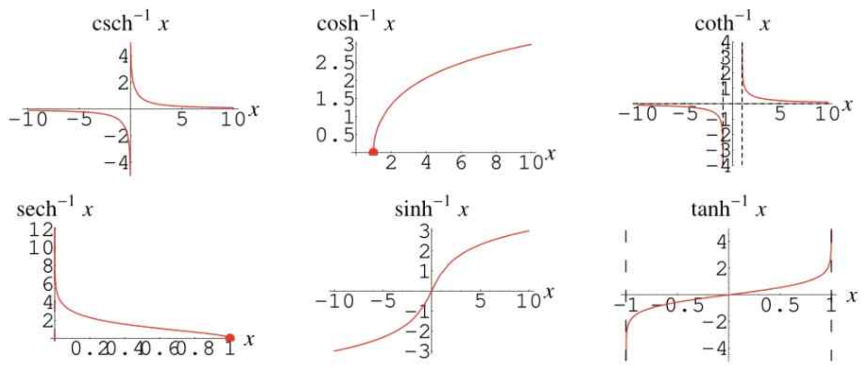
◎ Thm> $\operatorname{csch}^{-1}x = \ln\left(\dfrac{1+\sqrt{x^2+1}}{x}\right)$, $\operatorname{sech}^{-1}x = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right)$,

$\coth^{-1}x = \dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right)$

◎ Thm> $\dfrac{d}{dx}(\sinh^{-1}x) = \dfrac{1}{\sqrt{x^2+1}}$ $\qquad$ $\dfrac{d}{dx}(\operatorname{csch}^{-1}x) = -\dfrac{1}{|x|\sqrt{1+x^2}}$

$\dfrac{d}{dx}(\cosh^{-1}x) = \dfrac{1}{\sqrt{x^2-1}}$ $\qquad$ $\dfrac{d}{dx}(\operatorname{sech}^{-1}x) = -\dfrac{1}{x\sqrt{1-x^2}}$

$\dfrac{d}{dx}(\tanh^{-1}x) = \dfrac{1}{1-x^2}$ $\qquad$ $\dfrac{d}{dx}(\coth^{-1}x) = \dfrac{1}{1-x^2}$



§ 6.8 Indeterminate Forms and L'Hospital's Rule

◎ Thm> Cauchy's Mean Value Theorem : For functions f, g that are continuous on the interval $[a, b]$ and differentiable on the interval (a, b), and $g'(x) \neq 0$ for $\forall x \in (a, b)$, $\exists c \in (a, b)$ s.t. $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$.

pf) Let $h(x) := \dfrac{f(b)-f(a)}{g(b)-g(a)}(g(x)-g(a)) - (f(x)-f(a))$. Since $h \in C[a, b]$, $h \in D(a, b)$ and $h(a) = h(b) = 0$, $\exists c \in (a, b)$ s.t. $0 = h'(c) = \dfrac{f(b)-f(a)}{g(b)-g(a)}g'(c) - f'(c)$ by Rolle's Theorem. Therefore, $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$. ■

◎ Thm> L'Hospital's Rule : For differentiable functions f, g and $g'(x) \neq 0$ on an open interval I that contains a, if $\lim_{x\to a} f(x) = p$ and $\lim_{x\to a} g(x) = q$ $(p, q \in \{0, \pm\infty\})$ then $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \lim_{x\to a}\dfrac{f'(x)}{g'(x)}$ if the RHS limit exists.

pf) Let $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$. We will show that $\lim_{x \to a} \frac{f(x)}{g(x)} = L$. Since $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $0 < |x-a| < \delta \Rightarrow \left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon$.
Without loss of generalization, assume that $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$ and $f(a) = g(a) = 0$, then $f, g \in C[a, x]$ or $f, g \in C[x, a]$ and $f, g \in D(a, x)$ or $f, g \in D(x, a)$.

By Cauchy's Mean Value Theorem, $\exists c_1 \in (a, x)$ s.t.
$\frac{f(x)}{g(x)} = \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c_1)}{g'(c_1)}$, and $\exists c_2 \in (x, a)$ s.t.
$\frac{f(x)}{g(x)} = \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c_2)}{g'(c_2)}$. Since $0 < |c_1 - a| < \delta$ and $0 < |c_2 - a| < \delta$, we finally get $\left| \frac{f(x)}{g(x)} - L \right| < \varepsilon$ and $\lim_{x \to a} \frac{f(x)}{g(x)} = L$. ■

§ 7.1 Integration by Parts

◎ Thm> $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$ (IBP)

pf) $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$, $f(x)g'(x) = (f(x)g(x))' - f'(x)g(x)$
$\therefore \int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$ ■

ex) Reduction Formulas ($1 < n \in \mathbb{N}$)

$$\int \sin^n x dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x dx$$

$$\int \cos^n x dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx$$

$$\int \tan^n x dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x dx$$

$$\int (\ln x)^n dx = x(\ln x)^n - n \int (\ln x)^{n-1} dx$$

ex) $\int x^2 e^x dx = (x^2 - 2x + 2)e^x + C$

ex) $\int x^3 \sin x dx = ?$ (Fast IBP Method)

* Integration of Inverse Functions

$$\int f^{-1}(x)dx = \int yf'(y)dy = yf(y) - \int f(y)dy = xf^{-1}(x) - \int f(y)dy$$

$(y = f^{-1}(x),\ f(y) = x,\ f'(y)dy = dx)$

ex) $\int \sin^{-1}x dx = x\sin^{-1}x + \sqrt{1-x^2} + C$

ex) $\int \cos^{-1}x dx = x\cos^{-1}x - \sin(\cos^{-1}x) + C = x\cos^{-1}x - \sqrt{1-x^2} + C$ (by IBP)

ex) $\int \tan^{-1}x dx = x\tan^{-1}x + \ln|\cos(\tan^{-1}x)| + C$

§ 7.2 Trigonometric Integrals

◎ Thm> $\int \sin^m x \cos^n x dx$

(a) $n = 2k+1 \Rightarrow$ Take out one $\cos x$ and change the rest to terms of $\sin^2 x$.

(b) $m = 2k+1 \Rightarrow$ Take out one $\sin x$ and change the rest to terms of $\cos^2 x$.

(c) $m = 2k_1,\ n = 2k_2 \Rightarrow$ Use the half-angle formula : $\sin^2 x = \dfrac{1-\cos 2x}{2}$, $\cos^2 x = \dfrac{1+\cos 2x}{2}$.

<Example> $\int \cos^3 x dx = \int \cos x(1-\sin^2 x)dx = \int (\cos x - \sin^2 x\cos x)dx$
$= \sin x - \dfrac{1}{3}\sin^3 x + C$.

<Example> $\int \sin^5 x\cos^2 x dx = \int \sin x(1-\cos^2 x)^2\cos^2 x dx$
$= \int (\cos^2 x - 2\cos^4 x + \cos^6 x)\sin x dx = -\dfrac{1}{3}\cos^3 x + \dfrac{2}{5}\cos^5 x - \dfrac{1}{7}\cos^7 x + C$.

<Example> $\int \sin^4 x dx = \int \left(\dfrac{1-\cos 2x}{2}\right)^2 dx = \dfrac{1}{4}\int (1-2\cos 2x + \cos^2 2x)dx$
$= \dfrac{1}{4}\int \left(1-2\cos 2x + \dfrac{1+\cos 4x}{2}\right)dx = \dfrac{3}{8}x - \dfrac{1}{4}\sin 2x + \dfrac{1}{32}\sin 4x + C$.

◎ Thm> $\int \tan^m x\sec^n x dx$

(a) $n = 2k \Rightarrow$ Take out one $\sec^2 x$ and change the rest to terms of $\tan^2 x$.

(b) $m = 2k+1 \Rightarrow$ Take out one $\sec x\tan x$ and change the rest to terms of $\sec^2 x$.

<Example> $\int \tan^6 x \sec^4 x dx = \int \tan^6 x(1+\tan^2 x)\sec^2 x dx$

$= \frac{1}{7}\tan^7 x + \frac{1}{9}\tan^9 x + C.$

<Example> $\int \tan^5 x \sec^7 x dx = \int (\sec^2 x - 1)^2 \sec^6 x \sec x \tan x dx$

$= \int (\sec^{10} x - 2\sec^8 x + \sec^6 x)\sec x \tan x dx$

$= \frac{1}{11}\sec^{11} x - \frac{2}{9}\sec^9 x + \frac{1}{7}\sec^7 x + C.$

◎ Thm> $\int \tan x dx = \ln|\sec x| + C$

$\int \sec x dx = \ln|\sec x + \tan x| + C = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C = \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C$

§ 7.3 Trigonometric Substitution

◎ Thm> Trigonometric Substitutions

$\sqrt{a^2-x^2}$: $x = a\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 1-\sin^2\theta = \cos^2\theta$

$\sqrt{a^2+x^2}$: $x = a\tan\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 1+\tan^2\theta = \sec^2\theta$

$\sqrt{x^2-a^2}$: $x = a\sec\theta,\ 0 \le \theta < \frac{\pi}{2} \vee \pi \le \theta < \frac{3}{2}\pi,\ \sec^2\theta - 1 = \tan^2\theta$

§ 7.4 Integration of Rational Functions by Partial Fractions

◎ Thm> Partial Fractions

$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$ if $\deg P(x) \ge \deg Q(x)$.

(a) If $Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$, then

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}.$$

(b) If $Q(x) = (a_1x+b_1)^r(a_2x+b_2)\cdots(a_kx+b_k)$ $(2 \le r \in \mathbb{N})$, then

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + \cdots + \frac{A_r}{(a_1x+b_1)^r} + \frac{B_2}{a_2x+b_2} + \cdots + \frac{B_k}{a_kx+b_k}.$$

(c) If $Q(x) = (ax^2+bx+c)(a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$ where ax^2+bx+c cannot be factored into real coefficient polynomials ($b^2-4ac<0$), then

$$\frac{R(x)}{Q(x)} = \frac{Bx+C}{ax^2+bx+c} + \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}.$$

(d) If $Q(x) = (ax^2+bx+c)^r(a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$ where ax^2+bx+c cannot be factored into real coefficient polynomials ($b^2-4ac<0$) and $2 \le r \in \mathbb{N}$, then $\displaystyle\frac{R(x)}{Q(x)} = \sum_{i=1}^{r}\frac{A_ix+B_i}{(ax^2+bx+c)^i} + \sum_{i=1}^{k}\frac{C_i}{a_ix+b_i}$.

◎ cf) Heaviside Method

If $\displaystyle\frac{f(x)}{g(x)} = \frac{f(x)}{(x-a_1)(x-a_2)\cdots(x-a_n)} = \sum_{i=1}^{n}\frac{b_i}{x-a_i}$, then $\displaystyle b_i = \frac{f(a_i)}{h_i(a_i)}$ where $\displaystyle\frac{g(x)}{(x-a_i)} = h_i(x)$ for $1 \le i \le n$.

◎ Thm> Rationalization Substitution

<Example> $\displaystyle\int\frac{1}{x^2+x\sqrt{x}}dx = \int\frac{2u}{u^4+u^3}du = 2\int\frac{1}{u^2(u+1)}du$

$\left(u = \sqrt{x},\ du = \dfrac{1}{2\sqrt{x}}dx = \dfrac{1}{2u}dx\right)$

$$= 2\int\left(-\frac{1}{u}+\frac{1}{u^2}+\frac{1}{u+1}\right)du = -\frac{2}{u} - 2\ln u + 2\ln(u+1) + C$$

$$= -\frac{2}{\sqrt{x}} - \ln x + 2\ln(\sqrt{x}+1) + C$$

§ 7.5 Strategy for Integration

(a) Guidelines for Integration

(b) All continuous functions are integrable, but the antiderivatives of some functions cannot be expressed in terms of elementary functions. Elementary functions are the following group of functions : Polynomials, Rational Functions, Exponentials, Logarithms, Trigonometric Functions, Inverse Trigonometric Functions, Power Functions, Hyperbolic Functions, Inverse Hyperbolic Functions and other functions that can be obtained by operations of addition, subtraction, multiplication, division, and composition.

Although a derivative of an elementary function is an elementary function, it is not necessary for integration. (e.g. $\displaystyle\int e^{x^2}dx,\ \int\frac{e^x}{x}dx,\ \int\frac{\sin x}{x}dx,\ \int\sin(x^2)dx$)

§ 7.7 Approximate Integration

∗ Cases where it is impossible to compute the exact value of a definite integral
(a) The antiderivative cannot be expressed in terms of elementary functions
(b) The function itself is not identified exactly (especially in experiments or numerical analysis)

⇒ Use approximation methods to compute the value of the integral. (e.g. Midpoint Rule, Riemann Upper Sum, Riemann Lower Sum, Right Point Rule*, Left Point Rule*) (Methods involving partial sums) (*These are not official titles.)

◎ Thm> The Midpoint Rule

Consider a continuous function f defined on the interval $I=[a,b]$. When the interval $I=[a,b]$ is divided into n identical sub-intervals which all have the length of $\Delta x = \frac{b-a}{n}$, then $\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$ where $\Delta x = \frac{b-a}{n}$ and $x_i^* = \frac{x_{i-1}+x_i}{2}$. $\int_a^b f(x)dx = \left[f(x_1^*)+f(x_2^*)+\cdots+f(x_n^*)\right]\Delta x = M_n$.

◎ Thm> The Trapezoidal Rule

Consider a continuous function f defined on the interval $I=[a,b]$. When the interval $I=[a,b]$ is divided into n identical sub-intervals which all have the length of $\Delta x = \frac{b-a}{n}$, then $\int_a^b f(x)dx \approx \sum_{i=1}^{n} \frac{f(x_{i-1})+f(x_i)}{2}\Delta x$ where $\Delta x = \frac{b-a}{n}$. Rewriting the sum, we get

$$\int_a^b f(x)dx = \left[\frac{1}{2}f(x_0)+f(x_1)+f(x_2)+\cdots+f(x_{n-1})+\frac{1}{2}f(x_n)\right]\Delta x$$

$= T_n = \frac{L_n+R_n}{2}$ where L_n and R_n denotes the left point sum and the right point sum respectively.

◎ Thm> Simpson's Rule

The method replaces the integrand $f(x)$ by a quadratic polynomial (i.e. parabola) $P(x)$ that takes the same values as $f(x)$ at the end points a and b, and the midpoint $m = \frac{a+b}{2}$. Using the Lagrange polynomial interpolation, this polynomial can be expressed as :

$$P(x) = f(a)\frac{(x-m)(x-b)}{(a-m)(a-b)} + f(m)\frac{(x-a)(x-b)}{(m-a)(m-b)} + f(b)\frac{(x-a)(x-m)}{(b-a)(b-m)}.$$

Using integration by substitution, it can be shown that

$$\int_a^b P(x)dx = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \text{ where } h = \frac{b-a}{2} \text{ is the step size.}$$

Now break up the interval $[a, b]$ into n sub-intervals with the same length, where n is an even number. Then, the composite Simpson's rule is given by

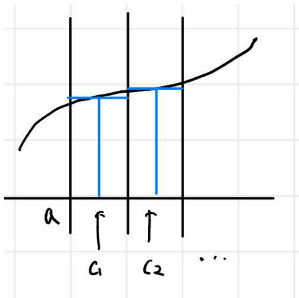
$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 4y_{n-1} + y_n) = S_n \text{ where}$$

$\Delta x = \dfrac{b-a}{n}$ and y_i $(0 \le i \le n)$ are the boundary points of each sub-interval. Using the Σ notation, we can also write as :

$$S_n = \frac{\Delta x}{3}\left(y_0 + y_n + 4\sum_{i=1}^{n/2} y_{2i-1} + 2\sum_{i=1}^{n/2} y_{2i}\right).$$

◎ Thm> If f, f', f'' are continuous on $[a, b]$ and M_n is the n-midpoint approximation, then $\exists\, c \in [a, b]$ s.t. $\displaystyle\int_a^b f(x)dx - M_n = \frac{(b-a)h^2}{24} f''(c)$.

pf)



For $k = 1, 2, 3, \cdots, n$, let $c_k = a + \left(k - \frac{1}{2}\right)h$ where $h = \frac{b-a}{n}$. Define function $\phi_k : \left[0, \frac{1}{2}h\right] \to \mathbb{R}$ such that $\phi_k(t) = \displaystyle\int_{c_k - t}^{c_k + t} f(x)dx - f(c_k)\cdot 2t$ $\left(t \in \left[0, \frac{1}{2}h\right]\right)$

For example, $\phi_k(0) = \displaystyle\int_{c_k}^{c_k} f(x)dx - f(c_k)\cdot 2\cdot 0 = 0$. Meanwhile,

$\phi_k(t) = \displaystyle\int_{c_k}^{c_k+t} f(x)dx - \int_{c_k}^{c_k-t} f(x)dx - f(c_k)\cdot 2t$ and

$\phi_k'(t) = f(c_k + t) + f(c_k - t) - 2f(c_k)$, $\phi_k''(t) = f'(c_k + t) - f'(c_k - t)$.

By the MVT, $\exists\, c_{k,t}$ s.t. $\phi_k''(t) = 2tf''(c_{k,t})$ and $|c_k - c_{k,t}| < t$.

$A = \inf\{f''(x) \mid x \in [a, b]\}$, $B = \sup\{f''(x) \mid x \in [a, b]\}$ then

$2tA \le \phi_k''(t) \le 2tB$, $t^2 A \le \phi_k'(t) \le t^2 B$, $\frac{1}{3}t^3 A \le \phi_k(t) \le \frac{1}{3}t^3 B$.

Since $t \in \left[0, \frac{1}{2}h\right]$, let $t = \frac{h}{2}$ then $\frac{1}{24}Ah^3 \le \phi_k\left(\frac{h}{2}\right) \le \frac{1}{24}Bh^3$

$$\frac{1}{24}Ah^3n = \frac{1}{24}A\frac{(b-a)^3}{n^2} \le \sum_{k=1}^{n}\phi_k\left(\frac{h}{2}\right) = \int_a^b f(x)dx - M_n \le \frac{1}{24}B\frac{(b-a)^3}{n^2} = \frac{1}{24}Bh^3n$$

$\dfrac{(b-a)^3A}{24n^2} \le \displaystyle\int_a^b f(x)dx - M_n \le \dfrac{(b-a)^3B}{24n^2}$. By the MVT, $\exists\, r \in [a,\, b]$ s.t.

$$\int_a^b f(x)dx - M_n = \frac{(b-a)^3}{24n^2}f''(r).$$

Corollary) Let $a \le x \le b$, $|f''(x)| \le k$ then $|E_M| \le \dfrac{k(b-a)^3}{24n^2}$.

◎ Thm> If f, f', f'' are continuous on $[a,\, b]$ and T_n is the Trapezoidal approximation, then $\exists\, c \in [a,\, b]$ s.t. $T_n - \displaystyle\int_a^b f(x)dx = \dfrac{(b-a)h^2}{12}f''(c)$. $(h = \dfrac{b-a}{n})$

pf) $k = 1, 2, \cdots, n$, $a_k = a + (k-1)h$, $t \in [0,\, h]$.

For function $\phi_k : [0,\, h] \to \mathbb{R}$, $\phi_k(t) = \dfrac{1}{2}t[f(a_k) + f(a_k+t)] - \displaystyle\int_{a_k}^{a_k+t} f(x)dx$,

$\phi_k(0) = 0$.

$\phi_k{}'(t) = \dfrac{1}{2}[f(a_k) + f(a_k+t)] + \dfrac{1}{2}tf'(a_k+t) - f(a_k+t)$

$= \dfrac{1}{2}[f(a_k) - f(a_k+t)] + \dfrac{1}{2}tf'(a_k+t)$, $\phi_k{}'(0) = 0$

$\phi_k{}''(t) = -\dfrac{1}{2}f'(a_k+t) + \dfrac{1}{2}f'(a_k+t) + \dfrac{1}{2}tf''(a_k+t) = \dfrac{1}{2}tf''(a_k+t)$,

$A = \inf\{f''(x) \mid x \in [a,\, b]\}$, $B = \sup\{f''(x) \mid x \in [a,\, b]\}$, $t \in [0,\, h]$

$\dfrac{1}{2}At \le \phi_k{}''(t) \le \dfrac{1}{2}Bt$, $\dfrac{1}{4}At^2 \le \phi_k{}'(t) \le \dfrac{1}{4}Bt^2$, $\dfrac{1}{12}At^3 \le \phi_k(t) \le \dfrac{1}{12}Bt^3$

$t = h$, $\dfrac{1}{12}Ah^3 \le \phi_k(h) \le \dfrac{1}{12}Bh^3$, $\displaystyle\sum_{k=1}^{n}\phi_k(h) = T_n - \int_a^b f(x)dx$ then

$$\frac{A(b-a)^3}{12n^2} \le T_n - \int_a^b f(x)dx \le \frac{B(b-a)^3}{12n^2}$$

Corollary) Let $a \le x \le b$, $|f''(x)| \le k$ then $|E_T| \le \dfrac{k(b-a)^3}{12n^2}$.

◎ Thm> If f, f', f'' are continuous on $[a,\, b]$ and S_n is the nth Simpson approximation, then $\exists\, c \in [a,\, b]$ s.t. $S_n - \displaystyle\int_a^b f(x)dx = \dfrac{(b-a)h^4}{180}f^{(4)}(c)$. $(h = \dfrac{b-a}{n})$

pf) For $k = 0, 1, 2, \cdots, \dfrac{1}{2}n - 1$, $x_n = a + (2k+1)h$, define function

$\phi_k : [0, h] \to \mathbb{R}$ as $\phi_k(t) = \frac{1}{3}t\left[f(x_k - t) + 4f(x_k) + f(x_k + t)\right] - \int_{x_k - t}^{x_k + t} f(x)dx$

and $\phi_k(0) = 0$. Then

$\phi_k{}'(t) = -\frac{2}{3}f(x_k - t) + \frac{4}{3}f(x_k) - \frac{2}{3}f(x_k + t) + \frac{1}{3}t(-f'(x_k - t) + f'(x_k + t))$,

$\phi_k{}'(0) = 0$, and

$\phi_k{}''(t) = \frac{1}{3}f'(x_k - t) - \frac{1}{3}f'(x_k + t) + \frac{1}{3}t(f''(x_k - t) + f''(x_k + t))$,

$\phi_k{}'''(t) = \frac{1}{3}t\left[f'''(x_k + t) - f'''(x_k - t)\right]$. By the MVT, $\exists\, c_{k,t}$ s.t.

$\phi_k{}'''(t) = \frac{1}{3}t \cdot 2t \cdot f^{(4)}(c_{k,t})$ and $|c_{k,t} - x_k| \le t$.

Let $A = \inf\{f^{(4)}(x) \mid x \in [a, b]\}$, $B = \sup\{f^{(4)}(x) \mid x \in [a, b]\}$, then

$\frac{2}{3}t^2 A \le \phi_k{}'''(t) \le \frac{2}{3}t^2 B$. For $\forall t \in [0, h]$, $\frac{1}{90}At^5 \le \phi_k(t) \le \frac{1}{90}Bt^5$, and

$t = h$ then $\frac{1}{90}Ah^5 \le \phi_k(h) \le \frac{1}{90}Bh^5$ and

$\frac{A(b-a)^5}{180n^4} \le \sum_{k=1}^{n/2-1} \phi_k(h) = S_n - \int_a^b f(x)dx \le \frac{B(b-a)^5}{180n^4}$. By the MVT,

$\exists\, c \in [a, b]$ s.t. $S_n - \int_a^b f(x)dx = \frac{(b-a)^5}{180n^4}f^{(4)}(c)$.

Corollary) Let $a \le x \le b$, $|f^{(4)}(x)| \le k$ then $|E_S| \le \frac{k(b-a)^5}{180n^4}$.

§ 7.8 Improper Integrals

Def> If $\int_a^t f(x)dx$ exists for $t \ge a$ and $\lim_{t\to\infty}\int_a^t f(x)dx$ exists, then

$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$ and it converges.

Def> If $\int_t^b f(x)dx$ exists for $t \le b$ and $\lim_{t\to-\infty}\int_t^b f(x)dx$ exists, then

$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx$ and it converges.

Def> If both $\int_a^{\infty} f(x)dx$ and $\int_{-\infty}^{a} f(x)dx$ exists, then

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^{\infty} f(x)dx + \int_{-\infty}^{0} f(x)dx.$$

ex) $\int_1^{\infty} \frac{1}{x}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}dx = \lim_{t\to\infty}[\ln|x|]_1^t = \lim_{t\to\infty}\ln|t| = \infty$: It is divergent.

ex) $\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \int_0^{\infty}\frac{1}{1+x^2}dx + \int_{-\infty}^{0}\frac{1}{1+x^2}dx$

$= \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}dx + \lim_{a\to-\infty}\int_a^0 \frac{1}{1+x^2}dx = \lim_{b\to\infty}[\tan^{-1}x]_0^b + \lim_{a\to-\infty}[\tan^{-1}x]_a^0$

$= \lim_{b\to\infty}\tan^{-1}b - \lim_{a\to-\infty}\tan^{-1}a = \pi$

ex) $\int_1^{\infty}\frac{1}{x^p}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}dx = \lim_{b\to\infty}\left[\frac{1}{1-p}x^{1-p}\right]_1^b = \lim_{b\to\infty}\left(\frac{b^{1-p}}{1-p} - \frac{1}{1-p}\right)$ $(p\neq 1)$

If $p=1$, $\int_1^{\infty}\frac{1}{x}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}dx = \lim_{b\to\infty}[\ln|x|]_1^b = \lim_{b\to\infty}\ln|b| = \infty$.

The integral converges if $p>1$ and diverges if $p\le 1$. This is called the p-test for infinite limits.

Def> Let function f be continuous on $[a, b)$ and discontinuous on b. Then,

$$\int_a^b f(x)dx = \lim_{t\to b}\int_a^t f(x)dx.$$

Def> Let function f be continuous on $(a, b]$ and discontinuous on a. Then,

$$\int_a^b f(x)dx = \lim_{t\to a}\int_t^b f(x)dx.$$

Def> Let function be discontinuous on c. If $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ exists, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

ex) For $0 < x \le 1$, let $f(x) = \frac{1}{\sqrt{x}}$ then $\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{t\to 0+}\int_t^1 \frac{1}{\sqrt{x}}dx = \lim_{t\to 0+}[2\sqrt{t}]_t^1$

$= \lim_{t\to 0+}2(1-\sqrt{t}) = 2$

ex) $\int_0^1 \ln x dx = \lim_{t\to 0+}\int_t^1 \ln x dx = \lim_{t\to 0+}[t\ln t - t]_t^1 = \lim_{t\to 0+}[-1 - t\ln t - t] = -1$

sukita1729@gmail.com

◎ Thm> Comparison Test

For continuous functions f, g and $x \geq a$, $f(x) \geq g(x) \geq 0$,

(a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.

(b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

ex) $\int_0^\infty e^{-x^2}dx = \int_0^1 e^{-x^2}dx + \int_1^\infty e^{-x^2}dx$ converges since $e^{-x^2} \leq e^{-x}$ for $x \geq 1$ and

$\int_1^\infty e^{-x}dx = \lim_{t\to\infty}[-e^{-t}]_1^t = \lim_{t\to\infty}(e^{-1} - e^{-t}) = e^{-1}$ converges.

§ 8.1 Arc Length

◎ Arc length of a curve

$$L = \lim_{n\to\infty}\sum_{i=1}^n |\mathrm{P}_{i-1}\mathrm{P}_i| = \lim_{n\to\infty}\sum_{i=1}^n \sqrt{(x_{i-1}-x_i)^2 + (y_{i-1}-y_i)^2} = \lim_{n\to\infty}\sum_{i=1}^n \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= \lim_{n\to\infty}\sum_{i=1}^n \sqrt{(\Delta x)^2 + \left[f'(x_i^*)\Delta x\right]^2} = \lim_{n\to\infty}\sum_{i=1}^n \Delta x\sqrt{1+\left[f'(x_i^*)\right]^2} \quad \text{by the MVT.}$$

Therefore, $L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ when f' is continuous on $[a, b]$ and thus integrable.

If a curve has the equation $x = g(y)$, $c \leq y \leq d$, and $g'(y)$ is continuous, then by interchanging the roles of x and y, we obtain the following formula for its length.

$$L = \int_c^d \sqrt{1+[g'(y)]^2}\,dy = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

ex) $f(x) = \frac{x^3}{12} + \frac{1}{x}$, $1 \leq x \leq 4$

$f'(x) = \frac{x^2}{4} - \frac{1}{x^2}$, $s(x) = \int_1^x \sqrt{1+[f'(t)]^2}\,dt = \int_1^x \left(\frac{t^2}{4} + \frac{1}{t^2}\right)dt = \left[\frac{t^3}{12} - \frac{1}{t}\right]_1^x$

$= \frac{x^3}{12} - \frac{1}{x} + \frac{11}{12}$

§ 8.2 Area of a Surface of Revolution

◎ Area of a surface of revolution

$$S = \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi \times \frac{f(x_{i-1})+f(x_i)}{2}\,|\mathrm{P}_{i-1}\mathrm{P}_i| = \lim_{n\to\infty}\sum_{i=1}^{n}\pi[f(x_{i-1})+f(x_i)]\sqrt{1+[f'(x_i^*)]^2}\,\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+[f'(x_i^*)]^2}\,\Delta x = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx = \int_a^b 2\pi y\,ds$$

ex) What is the area of the surface obtained by rotating $y = 2\sqrt{x}$ $(1 \le x \le 2)$ about the x-axis?

sol) $S = \int_1^2 2\pi y\sqrt{1+[f'(x)]^2}\,dx = \int_1^2 4\pi\sqrt{x}\sqrt{1+\frac{1}{x}}\,dx = \frac{8}{3}\pi(3\sqrt{3}-2\sqrt{2})$

§ 8.5 Probability

◎ Probability of continuous random variables

For probability density function f, $\mathrm{P}(a \le X \le b) = \int_a^b f(x)dx$,

$$E(X) = \int_a^b xf(x)dx = m,\ V(X) = \int_a^b (x-m)^2 f(x)dx = \int_a^b x^2 f(x)dx - \left(\int_a^b xf(x)dx\right)^2$$

(Q) Probability density function of the Normal Distribution

The probability density function : $f(x) = e^{-ax^2}$ such that $\int_{-\infty}^{\infty} f(x)dx = 1$,

$\int_{-\infty}^{\infty} xf(x)dx = m$, $\int_{-\infty}^{\infty}(x-m)^2 f(x)dx = V(X)$, $f(m-x) = f(m+x)$,

$f(x) \le f(m)$ for $\forall x \in \mathbb{R}$.

$f(x) = be^{-ax^2}$,

① $\int_{-\infty}^{\infty} f(x)dx = 1$

② $\int_{-\infty}^{\infty} xf(x)dx = 0$

③ $\int_{-\infty}^{\infty} x^2 f(x)dx = 1$

④ $f(m-x) = f(m+x)$

⑤ $f(m) = \max f$

Thm> $\int_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$

pf) $\int_{-\infty}^{\infty} e^{-ax^2}dx = \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a(x^2+y^2)}dydx\right]^{\frac{1}{2}} = \left[\int_{0}^{2\pi}\int_{0}^{\infty} e^{-ar^2}rdrd\theta\right]^{\frac{1}{2}}$

$= \left[2\pi\left[-\frac{1}{2a}e^{-ar^2}\right]_0^{\infty}\right]^{\frac{1}{2}} = \sqrt{\frac{\pi}{a}}$ ■

$\Rightarrow f(x) = \sqrt{\frac{a}{\pi}}e^{-ax^2}.$

Thm> $\sigma^2 = \int_{-\infty}^{\infty}(x-m)^2 f(x)dx,\ f(x) = \sqrt{\frac{a}{\pi}}e^{-ax^2},\ a = \frac{1}{2\sigma^2}$

pf) $m = 0,\ \sigma^2 = \sqrt{\frac{a}{\pi}}\int_{-\infty}^{\infty} x^2 e^{-ax^2}dx = \sqrt{\frac{a}{\pi}} \times \frac{1}{2a}\sqrt{\frac{\pi}{a}} = \frac{1}{2a}$ ■

$\therefore f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-m)^2}{2\sigma^2}\right),\ z = \frac{x-m}{\sigma}\ \Rightarrow f(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$