정적분 50題

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(1) 각변환

$\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = \cos\theta$ $\tan(-\theta) = -\tan\theta$

$\sin(\pi+\theta) = -\sin\theta$ $\cos(\pi+\theta) = -\cos\theta$ $\tan(\pi+\theta) = \tan\theta$

$\sin(\pi-\theta) = \sin\theta$ $\cos(\pi-\theta) = -\cos\theta$ $\tan(\pi-\theta) = -\tan\theta$

$\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$ $\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$ $\tan\left(\frac{\pi}{2}+\theta\right) = -\cot\theta$

$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$ $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$ $\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$

$\sin\left(\frac{3}{2}\pi+\theta\right) = -\cos\theta$ $\cos\left(\frac{3}{2}\pi+\theta\right) = \sin\theta$ $\tan\left(\frac{3}{2}\pi+\theta\right) = -\cot\theta$

$\sin\left(\frac{3}{2}\pi-\theta\right) = -\cos\theta$ $\cos\left(\frac{3}{2}\pi-\theta\right) = -\sin\theta$ $\tan\left(\frac{3}{2}\pi-\theta\right) = \cot\theta$

(2) 덧셈정리

$$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta\pm\cos\alpha\sin\beta$$

$$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta\mp\sin\alpha\sin\beta$$

$$\tan(\alpha\pm\beta) = \frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta} \quad (1\mp\tan\alpha\tan\beta\neq 0)$$

(3) 배각공식

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = \cos^2\alpha-\sin^2\alpha = 2\cos^2\alpha-1 = 1-2\sin^2\alpha$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} \quad (1-\tan^2\alpha\neq 0)$$

$$\sin 3\alpha = 3\sin\alpha-4\sin^3\alpha$$

$$\cos 3\alpha = 4\cos^3\alpha-3\cos\alpha$$

$$\tan 3\alpha = \frac{3\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha} \quad (1-3\tan^2\alpha\neq 0)$$

(4) 반각공식

$$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}$$

$$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}$$

$$\tan^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{1+\cos\alpha}$$

(5) 곱을 합 또는 차로 고치는 공식

$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$

$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$

$\sin\alpha\sin\beta = -\frac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha-\beta)]$

(6) 합 또는 차를 곱으로 고치는 공식

$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

(7) 삼각함수의 합성

$a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}\sin(\theta+\alpha) = \sqrt{a^2+b^2}\cos(\theta-\beta)$

$\alpha = \tan^{-1}\left(\frac{b}{a}\right),\ \beta = \tan^{-1}\left(\frac{a}{b}\right) = \frac{\pi}{2} - \alpha$

(8) 삼각함수 항등식

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

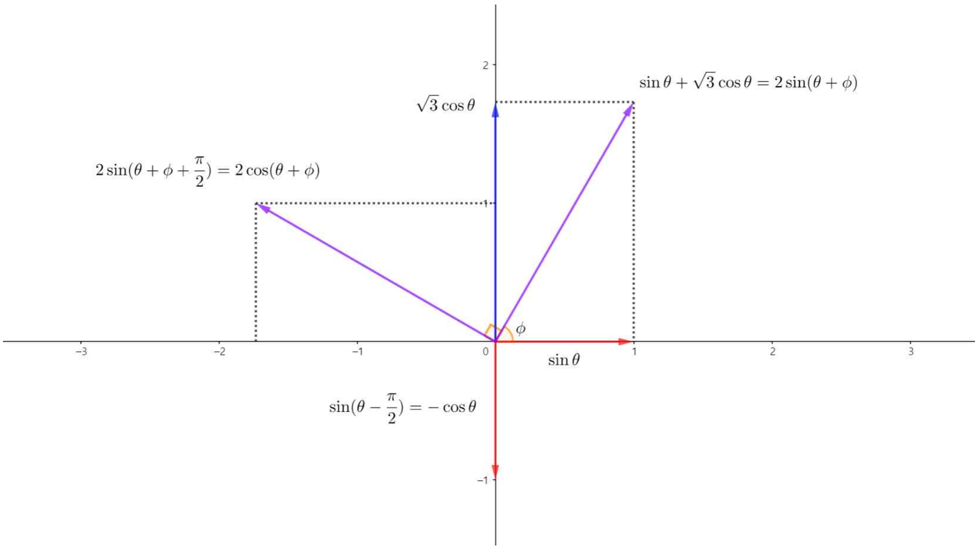
$\cot^2\theta + 1 = \csc^2\theta$

(9) 부정적분과 미분

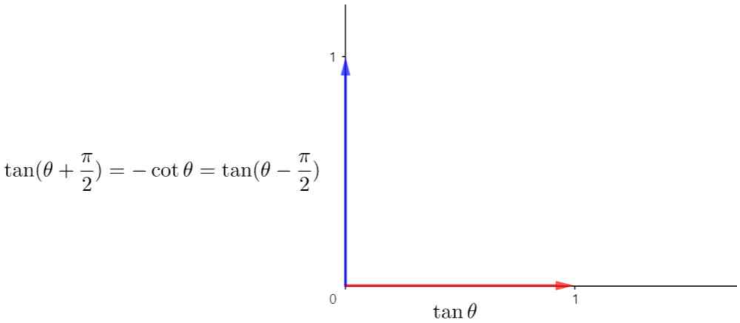
① $\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$

② $\int\left(\frac{d}{dx}f(x)\right)dx = f(x) + C$

(10) 삼각함수 위상자(Phasor)



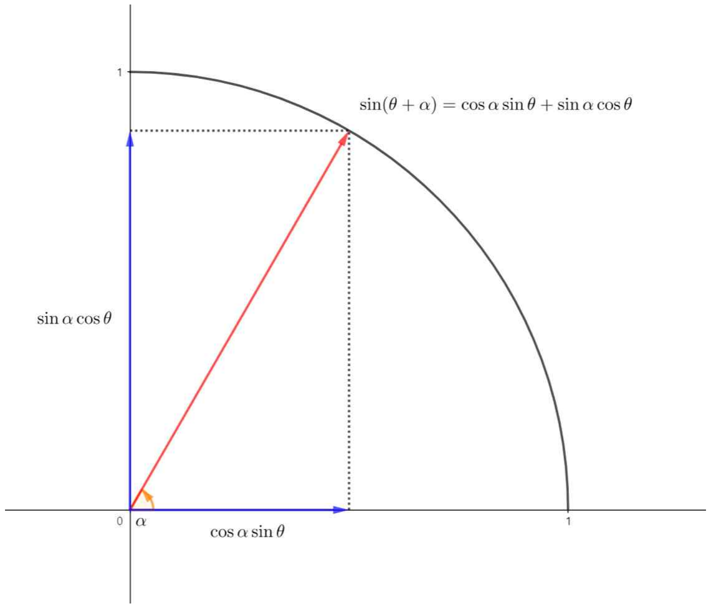
x축의 양의 방향을 $\sin$축, y축의 양의 방향을 $\cos$축으로 설정하여 삼각함수의 위상을 벡터로 표현한다. 각이 더해질 경우 위상자는 길이는 유지된 채 반시계방향으로 회전한다.
위상자가 표현하는 삼각함수의 계수는 그 위상자의 길이로 표현된다. 서로 다른 두 위상자를 벡터합하면 이는 각 위상자가 표현하는 삼각함수의 합성과 같다. 가령, 위 사진에서 $\sin\theta+\sqrt{3}\cos\theta=2\sin(\theta+\phi)$이고 $\phi=\frac{\pi}{3}$이다. $\sin\theta$ 위상자를 시계방향으로 $\frac{\pi}{2}$ 만큼 회전시키면 $\sin\left(\theta-\frac{\pi}{2}\right)$이며, 이는 $-\cos$축이므로 $\sin\left(\theta-\frac{\pi}{2}\right)=-\cos\theta$이다. $\sin$을 $\csc$로, $\cos$을 $\sec$로 바꾸면 $\csc$와 $\sec$에 대한 각변환이 가능하나 덧셈정리와 합성은 성립하지 않는다. 즉, $\csc\theta+\sqrt{3}\sec\theta=2\csc(\theta+\phi)$는 성립하지 않는다.



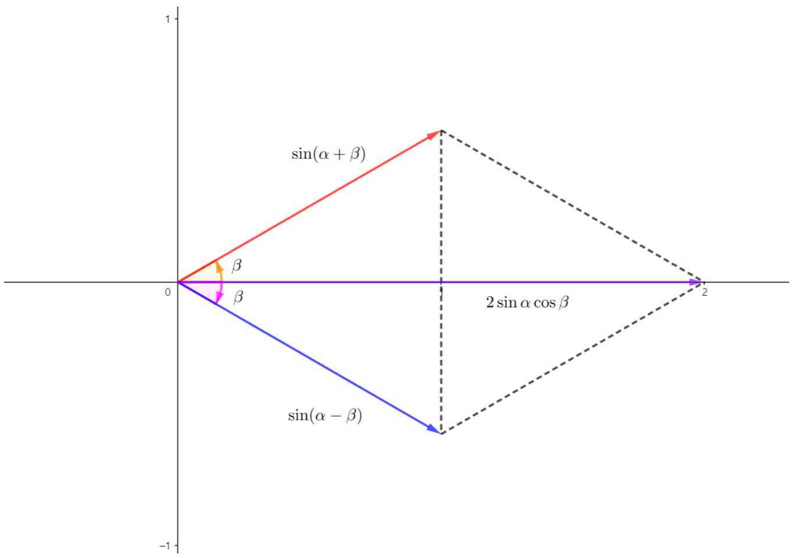
$\tan$와 $-\cot$의 경우 위와 같이 xy평면의 제 1사분면만을 이용하여 도시할 수 있다. 이 경우 주기가 π이므로 시계방향으로 회전하는 경우 돌아간 각도를 $\frac{3}{2}\pi$가 아닌 $\frac{\pi}{2}$로 본다.

(11) 삼각함수 위상자의 활용

삼각함수 위상자를 사용하면 (1), (2), (5), (6), (7)의 공식들은 모두 증명 가능하다. 이에 대한 예시로 (2)와 (5)의 첫 번째 공식의 증명을 제시해 놓는다.



그림과 같은 단위원에서 $\sin(\theta+\alpha)$는 $\sin\theta$축에서 길이 1인 위상자가 각도 α만큼 회전한 것이다. 따라서 위상자의 종점에서 sin축, cos축에 각각 수선의 발을 내리면 원점을 시점으로 하고 두 수선의 발을 종점으로 하는 두 위상자의 길이는 각각 $\cos\alpha$, $\sin\alpha$이다. 즉 처음의 $\sin(\theta+\alpha)$ 위상자(빨간색)를 sin축 성분과 cos축 성분의 두 위상자(파란색)로 분해할 수 있고 이들의 길이는 각각 $\cos\alpha$, $\sin\alpha$이므로, $\sin(\theta+\alpha) = \cos\alpha\sin\theta + \sin\alpha\cos\theta$가 성립한다.



그림과 같이 기준각이 α인 위상 평면에서 $\sin(\alpha+\beta)$, $\sin(\alpha-\beta)$는 길이가 1인 $\sin\alpha$축

위상자가 각각 β, $-\beta$만큼 회전한 것이다. 따라서 이들을 합성한 보라색 위상자는 $\sin\alpha$ 축으로 길이가 $2\cos\beta$인 위상자이므로 $\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$이고 증명이 완료되었다. 이와 비슷한 방법으로 (1), (2), (5), (6), (7)의 공식들을 모두 증명할 수 있다.

(12) 부정적분의 기본 성질

① $\int kf(x)dx = k\int f(x)dx \ (k\in\mathbb{R})$

② $\int [f(x)\pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$ (복부호동순)

(13) 치환적분법

① $g(x)=t$일 때, $\int f(g(x))g'(x)dx = \int f(t)dt$

② $\int f(x)dx = F(x)+C$일 때, $\int f(ax+b)dx = \frac{1}{a}F(ax+b)+C$

③ $\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b|+C$

④ $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)|+C$

(14) 바이어슈트라스 치환 (Weierstrass Substitution)

바이어슈트라스 치환은 삼각함수의 유리 적분을 유리식의 적분으로 바꿔주는 치환법이다.

$$\tan\frac{x}{2}=t$$

$$\sin x = \frac{2t}{1+t^2},\quad \cos x = \frac{1-t^2}{1+t^2},\quad \tan x = \frac{2t}{1-t^2}$$

(15) 오일러 치환 (Euler's Substitution)

오일러 치환은 유리 이변수 함수 R에 대하여 다음과 같은 부정적분을 계산하기 위한 치환법이다.

$$\int R(x,\ \sqrt{ax^2+bx+c}\,)dx$$

[1] 제 1종 오일러 치환

$a>0$일 때, 다음과 같이 치환한다.

$$\sqrt{ax^2+bx+c} = \pm x\sqrt{a}+t,\quad x=\frac{c-t^2}{\pm 2t\sqrt{a}-b}$$

와 같이 치환한다.

[2] 제 2종 오일러 치환

$c > 0$일 때, 다음과 같이 치환한다.

$$\sqrt{ax^2+bx+c} = xt \pm \sqrt{c}, \quad x = \frac{\pm 2t\sqrt{c}-b}{a-t^2}$$

[3] 제 3종 오일러 치환

방정식 $ax^2+bx+c=0$이 두 실근 α, β를 가질 때, 다음과 같이 치환한다.

$$\sqrt{ax^2+bx+c} = \sqrt{a(x-\alpha)(x-\beta)} = (x-\alpha)t, \quad x = \frac{\alpha\beta-\alpha t^2}{a-t^2}$$

(16) 부정적분 공식 ($C \in \mathbb{R}$)

1. $\int dx = x + C$
2. $\int a dx = ax + C \ (a \in \mathbb{R})$
3. $\int x^n dx = \frac{1}{n+1}x^{n+1} + C \ (-1 \neq n \in \mathbb{R})$
4. $\int \frac{1}{x} dx = \ln|x| + C$
5. $\int e^x dx = e^x + C$
6. $\int a^x dx = \frac{a^x}{\ln a} + C \quad (0 < a \neq 1)$
7. $\int \ln x dx = x\ln x - x + C$
8. $\int \sin x dx = -\cos x + C$
9. $\int \cos x dx = \sin x + C$
10. $\int \tan x dx = \ln|\sec x| + C$
11. $\int \csc x dx = \ln|\csc x - \cot x| + C$
12. $\int \sec x dx = \ln|\sec x + \tan x| + C$
13. $\int \cot x dx = \ln|\sin x| + C$
14. $\int \sec^2 x dx = \tan x + C$
15. $\int \csc^2 x dx = -\cot x + C$
16. $\int \sec x \tan x dx = \sec x + C$
17. $\int \csc x \cot x dx = -\csc x + C$

18. $\displaystyle\int \frac{1}{(x+a)(x+b)}dx = \frac{1}{b-a}\ln\left|\frac{x+a}{x+b}\right| + C$

19. $\displaystyle\int \frac{1}{\sqrt{x^2 \pm a^2}}dx = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$

(17) 헤비사이드 법 (부분분수 분해)

$\displaystyle\frac{g(x)}{f(x)} = \frac{g(x)}{(x-a_1)(x-a_2)\cdots(x-a_n)} = \sum_{i=1}^{n}\frac{b_i}{x-a_i} = \frac{b_1}{x-a_1} + \frac{b_2}{x-a_2} + \cdots + \frac{b_n}{x-a_n}$ 일 때

$\displaystyle\frac{f(x)}{(x-a_i)} = h_i(x)$라 하면 $\displaystyle b_i = \frac{g(a_i)}{h_i(a_i)}$가 성립한다. $(1 \le i \le n)$

(18) 부분적분법

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

(19) 삼각함수의 거듭제곱의 부정적분 공식 (Reduction Formula)

$n \in \mathbb{N} - \{1\}$,

$$\int \sin^n x dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x dx$$

$$\int \cos^n x dx = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x dx$$

$$\int \tan^n x dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x dx$$

(20) 정적분 테크닉

① 적분 구간을 이용한 치환 $(x \mapsto a+b-x)$

$$\int_a^b f(x)dx = \int_b^a f(a+b-x)(-dx) = \int_a^b f(a+b-x)dx$$

$f(x)+f(a+b-x)$의 정적분이 쉽게 계산되는 경우

$$\int_a^b f(x)dx = \int_a^b \frac{f(x)+f(a+b-x)}{2}dx$$

임을 이용하여 정적분을 계산할 수 있다.

② 대칭 치환 $(x \mapsto -x)$

$$\int_{-c}^{c} f(x)dx = \int_{c}^{-c} f(-x)(-dx) = \int_{-c}^{c} f(-x)dx$$

이는 ①에서 $a+b=0$인 특수한 경우이지만, 마찬가지로 자주 등장한다.

$$\int_{-c}^{c} f(x)dx = \int_{-c}^{c} \frac{f(x)+f(-x)}{2}dx$$

※ 고등학교 미적분 수준을 벗어나는 문제는 없습니다. 모든 문제는 고등학교 미적분 교과 내용만을 이용하여 해결할 수 있습니다.

※ 모든 문제는 정확한 수치를 구할 수 있습니다.

[1 ~ 50] 다음 정적분의 값을 구하시오.

1. $\int_{-1}^{1} \frac{e^{3x}-e^{-3x}}{\cos x}dx$

2. $\int_{0}^{\frac{\pi}{3}} \frac{\sin 2x}{2+\cos x}dx$

3. $\int_{0}^{\pi} \frac{x\sin x}{3+\sin^2 x}dx$

4. $\int_{0}^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}dx$

5. $\int_{0}^{\frac{\pi}{2}} \frac{\cos^{\pi} x}{\sin^{\pi} x+\cos^{\pi} x}dx$

6. $\int_{0}^{\frac{\pi}{4}} \frac{\sin x}{\sin x+\cos x}dx$

7. $\int_{-1}^{1} \frac{3x^2}{e^x+1}dx$

8. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{e^{1/x}+1}dx$

9. $\int_{0}^{1} \frac{1}{1+\left(1-\frac{1}{x}\right)^{2022}}dx$

10. $\int_{0}^{\frac{\pi}{2}} \sin^{10} x dx$

11. $\int_{0}^{1} \sqrt[3]{2x^3-3x^2-x+1}\,dx$

12. $\int_{0}^{1} \frac{\ln(x+1)}{x^2+1}dx$

13. $\int_{0}^{1} \sqrt{x(1-x)}\,dx$

14. $\displaystyle\int_0^{\frac{3}{4}} \sqrt{x}\,\sqrt{1-x}\,dx$

15. $\displaystyle\int_{-1}^{3} \frac{x^2-2x+3}{e^{2x-2}+1}\,dx$

16. $\displaystyle\int_1^{e} \frac{x^4+81}{x(x^2+9)^2}\,dx$

17. $\displaystyle\int_0^{1} x^e(\ln x)^{2022}dx$

18. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{\sin 2x+\cos 2x+1}\,dx$

19. $\displaystyle\int_{\sqrt[3]{3}}^{4} \sqrt{\frac{x^3-3}{x^{11}}}\,dx$

20. $\displaystyle\frac{2^{21}}{\pi}\int_0^{1} \frac{x^{20}}{\sqrt{1-x^2}}\,dx$

21. $\displaystyle\int_0^{\pi} \frac{x\tan x}{\sec x+\cos x}\,dx$

22. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\ln(1+e^x)dx$

23. $\displaystyle\int_0^{1} \frac{1-x^{99}}{(1+x)(1+x^{100})}\,dx$

24. $\displaystyle\int_0^{\frac{\pi}{8}} \frac{\tan 2x}{\sqrt{\sin^6 x+\cos^6 x}}\,dx$

25. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{e^x\tan x(\cos 2x-3)}{1+\cos 2x}\,dx$

26. $\displaystyle\int_2^{4} \frac{1}{x^{2022}-x}\,dx$

27. $\displaystyle\int_3^{9} \frac{1}{x(x^{\sqrt{e}}-1)}\,dx$

28. $\displaystyle\int_0^{\frac{\pi}{2}} \sin 2022x\cdot\sin^{2020}x\,dx$

29. $\displaystyle\int_1^{\sqrt{2}} \frac{x^4-1}{x^2\sqrt{x^4-x^2+1}}\,dx$

30. $\displaystyle\int_3^{4} \frac{x+4}{x^3+3x^2-10x}\,dx$

31. $\displaystyle\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sec^2 x}{(\sec x+\tan x)^{3/2}}\,dx$

32. $\displaystyle\int_0^{\frac{\pi}{2}} \ln(\cos x)dx$

33. $\int_{1}^{\sqrt[6]{6}} \frac{6}{x(x^6+2)^2}\,dx$

34. $\int_{0}^{1} \frac{x^2}{x+\sqrt{1-x^2}}\,dx$

35. $\int_{0}^{\frac{\pi}{4}} \frac{1}{1-3\cos^2 x}\,dx$

36. $\int_{0}^{\frac{1}{2}} \frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}}\,dx$

37. $\int_{1}^{\frac{\pi}{2}} x^{\sin x-1}(x\cos x\ln x+\sin x)dx$

38. $\int_{0}^{1}\left(\sqrt[3]{1-x^7}-\sqrt[7]{1-x^3}\right)dx$

39. $\int_{0}^{2}(\sqrt{1+x^3}+\sqrt[3]{x^2+2x})dx$

40. $\int_{0}^{\frac{\pi}{2}} \frac{\sin x}{1+\sqrt{\sin 2x}}\,dx$

41. $\int_{0}^{1}(\ln x)^{2022}dx$

42. $\int_{0}^{\frac{\pi}{2}} \cos^{999}x\cos 999x\,dx$

43. $\int_{0}^{\frac{\pi}{2}} \frac{x}{\tan x}\,dx$

44. $\int_{0}^{\frac{\pi}{12}} \sin^2\left(x-\sqrt{\frac{\pi^2}{12^2}-x^2}\right)dx$

45. $\int_{1}^{e^2} \frac{\ln(xe^{x+1})}{(x+1)^2+\left[\ln(x^x)\right]^2}\,dx$

46. $\int_{1}^{e} \frac{(x+1)^2+(3x+1)\sqrt{x+\ln x}}{2x\sqrt{x+\ln x}\,(x+\sqrt{x+\ln x})}\,dx$

47. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sqrt{\cos 2x}}{\sin x}\,dx$

48. $\int_{0}^{\pi}\ln(\pi^2-2\pi\cos x+1)dx$

49. $\int_{0}^{\pi}\ln(e\cos x+\pi)dx$

50. $\int_{0}^{1} \frac{1}{(1+x^{\phi})^{\phi}}\,dx, \quad \phi=\frac{1+\sqrt{5}}{2}$

[해설]

1. $I = \int_{-1}^{1} \frac{e^{3x} - e^{-3x}}{\cos x} dx = \int_{-1}^{1} \frac{e^{-3x} - e^{3x}}{\cos x} dx = -I$

$\therefore I = \int_{-1}^{1} \frac{e^{3x} - e^{-3x}}{\cos x} dx = 0$ ■

2. $u = \cos x,\ du = -\sin x dx$

$\int_{0}^{\frac{\pi}{3}} \frac{\sin 2x}{2+\cos x} dx = (-2)\int_{0}^{\frac{\pi}{3}} \frac{\cos x(-\sin x)}{2+\cos x} dx = (-2)\int_{1}^{\frac{1}{2}} \frac{u}{2+u} du = 2\int_{\frac{1}{2}}^{1}\left(1-\frac{2}{u+2}\right)du$

$= 2\left[u - 2\ln|u+2|\right]_{\frac{1}{2}}^{1} = 2(1-\ln 9) - 2\left(\frac{1}{2} - 2\ln\frac{5}{2}\right) = 1 - \ln\left(\frac{1296}{625}\right)$ ■

3. $I = \int_{0}^{\pi} \frac{x\sin x}{3+\sin^2 x} dx = \int_{0}^{\pi} \frac{(\pi - x)\sin x}{3+\sin^2 x} dx\ \ (x \mapsto \pi - x)$

$2I = \int_{0}^{\pi}\left(\frac{x\sin x}{3+\sin^2 x} + \frac{(\pi-x)\sin x}{3+\sin^2 x}\right)dx = \int_{0}^{\pi} \frac{\pi\sin x}{3+\sin^2 x} dx = \int_{0}^{\pi} \frac{\pi \sin x}{4-\cos^2 x} dx$

$= \int_{1}^{-1} \frac{\pi}{u^2-4} du\ \ (u = \cos x,\ du = -\sin x dx)$

$= \pi\left[\frac{1}{4}\ln\left|\frac{u-2}{u+2}\right|\right]_{1}^{-1} = \frac{\pi}{2}\ln 3$

$\therefore I = \frac{\pi}{4}\ln 3$ ■

4. $I = \int_{0}^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx,\ \frac{\pi}{2} - I = \int_{0}^{\frac{\pi}{2}} dx - \int_{0}^{\frac{\pi}{2}} \frac{2\sin x}{\sin x + \cos x} dx$

$= \int_{0}^{\frac{\pi}{2}} \frac{\cos x - \sin x}{\sin x + \cos x} dx = \left[\ln|\sin x + \cos x|\right]_{0}^{\frac{\pi}{2}} = 0$

$\therefore I = \frac{\pi}{4}$ ■

5. $I=\int_0^{\frac{\pi}{2}}\frac{\cos^{\pi}x}{\sin^{\pi}x+\cos^{\pi}x}dx=\int_0^{\frac{\pi}{2}}\frac{\sin^{\pi}x}{\sin^{\pi}x+\cos^{\pi}x}dx\ \left(x\mapsto\frac{\pi}{2}-x\right)$

$2I=\int_0^{\frac{\pi}{2}}\left(\frac{\cos^{\pi}x}{\sin^{\pi}x+\cos^{\pi}x}+\frac{\sin^{\pi}x}{\sin^{\pi}x+\cos^{\pi}x}\right)dx=\frac{\pi}{2}$

$\therefore I=\frac{\pi}{4}$ ■

6. $\int_0^{\frac{\pi}{4}}\frac{\sin x}{\sin x+\cos x}dx=\int_0^{\frac{\pi}{4}}\frac{\sin\left(\frac{\pi}{4}-x\right)}{\sin\left(\frac{\pi}{4}-x\right)+\cos\left(\frac{\pi}{4}-x\right)}dx\ \left(x\mapsto\frac{\pi}{4}-x\right)$

$=\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{\cos x-\sin x}{\cos x}dx=\frac{1}{2}\int_0^{\frac{\pi}{4}}(1-\tan x)dx=\frac{1}{2}\left[x+\ln|\cos x|\right]_0^{\frac{\pi}{4}}=\frac{\pi}{8}-\frac{1}{4}\ln 2$ ■

7. $I=\int_{-1}^{1}\frac{3x^2}{e^x+1}dx=\int_{-1}^{1}\frac{3x^2}{e^{-x}+1}dx=\int_{-1}^{1}\frac{e^x\cdot 3x^2}{e^x+1}dx\ (x\mapsto -x)$

$2I=\int_{-1}^{1}\left(\frac{3x^2}{e^x+1}+\frac{e^x\cdot 3x^2}{e^x+1}\right)dx=\int_{-1}^{1}x^2dx=\left[x^3\right]_{-1}^{1}=2$

$\therefore I=\int_{-1}^{1}\frac{3x^2}{e^x+1}dx=1$ ■

8. $I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos x}{e^{1/x}+1}dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos x}{e^{-1/x}+1}dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{e^{1/x}\cdot\cos x}{e^{1/x}+1}dx\ (x\mapsto -x)$

$2I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\cos x}{e^{1/x}+1}+\frac{e^{1/x}\cdot\cos x}{e^{1/x}+1}\right)dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x\,dx=2$

$\therefore I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos x}{e^{1/x}+1}dx=1$ ■

9. $I=\int_0^1\frac{1}{1+\left(1-\frac{1}{x}\right)^{2022}}dx=\int_0^1\frac{1}{1+\left(\frac{x-1}{x}\right)^{2022}}dx=\int_0^1\frac{x^{2022}}{x^{2022}+(x-1)^{2022}}dx$

$$= \int_0^1 \frac{(1-x)^{2022}}{x^{2022}+(x-1)^{2022}}\,dx \quad (x \mapsto 1-x)$$

$$= \frac{1}{2}\int_0^1 \left(\frac{x^{2022}}{x^{2022}+(1-x)^{2022}} + \frac{(1-x)^{2022}}{x^{2022}+(1-x)^{2022}} \right) dx = \frac{1}{2} \ \blacksquare$$

10. $W_n := \int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \sin^{n-1}x \cdot \sin x dx$

$$= \left[-\cos x \cdot \sin^{n-1}x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2}x\cos x \cdot (-\cos x)dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x\cos^2 x dx = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x(1-\sin^2 x)dx$$

$$= (n-1)(W_{n-2} - W_n)$$

$$nW_n = (n-1)W_{n-2}, \ W_n = \frac{n-1}{n}W_{n-2} \ (n \geq 2)$$

한편 $W_0 = \int_0^{\frac{\pi}{2}} 1dx = \frac{\pi}{2}$, $W_1 = \int_0^{\frac{\pi}{2}} \sin x dx = 1$이므로

$$W_n = \frac{n-1}{n}W_{n-2} = \frac{(n-1)(n-3)}{n(n-2)}W_{n-4} = \cdots$$

$$= \begin{cases} \dfrac{(n-1)(n-3)\cdots 1}{n(n-2)\cdots 2} \cdot W_0 & (n = 2m) \\ \dfrac{(n-1)(n-3)\cdots 2}{n(n-2)\cdots 1} \cdot W_1 & (n = 2m+1) \end{cases}$$

$$\therefore W_n = \frac{(n-1)!!}{n!!} \cdot \left(\frac{\pi}{2}\right)^{\frac{1+(-1)^n}{2}} \ (n > 0), \ W_0 = \frac{\pi}{2} \ \text{(Wallis' Formula)}$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^{10}x dx = W_{10} = \frac{9 \cdot 7 \cdot 5 \cdot 3 \cdot 1}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{63\pi}{512} \ \blacksquare$$

11. $I = \int_0^1 \sqrt[3]{2x^3 - 3x^2 - x + 1}\,dx = \int_0^1 \sqrt[3]{2(1-x)^3 - 3(1-x)^2 - (1-x) + 1}\,dx \ (x \mapsto 1-x)$

$$= \int_0^1 -\sqrt[3]{2x^3-3x^2-x+1}\,dx = -I$$

$$\therefore I = \int_0^1 \sqrt[3]{2x^3-3x^2-x+1}\,dx = 0 \blacksquare$$

12. $x = \tan t,\ dx = \sec^2 t dt,\ \dfrac{dx}{1+x^2} = dt$

$$I = \int_0^1 \frac{\ln(x+1)}{x^2+1}dx = \int_0^{\frac{\pi}{4}} \ln(1+\tan t)dt = \int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4}-t\right)\right)dt \ \left(t \mapsto \frac{\pi}{4}-t\right)$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan t}{1+\tan t}\right)dt = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan t}\right)dt = \int_0^{\frac{\pi}{4}} [\ln 2 - \ln(1+\tan t)]\,dt$$

$$2I = \int_0^{\frac{\pi}{4}} [\ln(1+\tan t)+(\ln 2-\ln(1+\tan t))]\,dt = \int_0^{\frac{\pi}{4}} \ln 2 dt = \frac{\pi}{4}\ln 2$$

$$\therefore I = \int_0^1 \frac{\ln(x+1)}{x^2+1}dx = \frac{\pi}{8}\ln 2 \blacksquare$$

13. $y = \sqrt{x(1-x)}$ 라 하면 $y^2 = x - x^2,\ x^2 - x + y^2 = 0$이므로 $\left(x-\dfrac{1}{2}\right)^2 + y^2 = \dfrac{1}{4}$이고 구하는 정적분은 점 $\left(\dfrac{1}{2}, 0\right)$을 중심으로 하고 반지름의 길이가 $\dfrac{1}{2}$인 반원의 넓이이다.

$$\therefore \int_0^1 \sqrt{x(1-x)}\,dx = \frac{\pi}{8} \blacksquare$$ (삼각치환을 통해 구할 수도 있다.)

14. $t = \sqrt{x},\ dt = \dfrac{1}{2\sqrt{x}}dx,\ t = \sin u,\ dt = \cos u du$

$$\int_0^{\frac{3}{4}} \sqrt{x}\sqrt{1-x}\,dx = \int_0^{\frac{\sqrt{3}}{2}} t\sqrt{1-t^2}\cdot 2t dt = 2\int_0^{\frac{\sqrt{3}}{2}} t^2\sqrt{1-t^2}\,dt$$

$$= 2\int_0^{\frac{\pi}{3}} \sin^2 u \cos u \cdot \cos u du = \frac{1}{2}\int_0^{\frac{\pi}{3}} \sin^2 2u du = \frac{1}{4}\int_0^{\frac{\pi}{3}} (1-\cos 4u)du$$

$$= \left[\frac{1}{4}u - \frac{1}{16}\sin 4u\right]_0^{\frac{\pi}{3}} = \frac{\sqrt{3}}{32} + \frac{\pi}{12} \blacksquare$$

15. $u = x-1,\ du = dx$

$$\int_{-1}^{3} \frac{x^2-2x+3}{e^{2x-2}+1}dx = \int_{-2}^{2} \frac{u^2+2}{e^{2u}+1}du = \int_{-2}^{2} \frac{u^2+2}{e^{-2u}+1}du \ (u \mapsto -u)$$

$$= \int_{-2}^{2} \frac{e^{2u}(u^2+2)}{e^{2u}+1}du = \frac{1}{2}\int_{-2}^{2}\left(\frac{u^2+2}{e^{2u}+1} + \frac{e^{2u}(u^2+2)}{e^{2u}+1}\right)du = \frac{1}{2}\int_{-2}^{2}(u^2+2)du$$

$$= \int_{0}^{2}(u^2+2)du = \left[\frac{1}{3}u^3+2u\right]_0^2 = \frac{20}{3} \blacksquare$$

16. $t = x^2,\ dt = 2xdx$

$$\int_{1}^{e} \frac{x^4+81}{x(x^2+9)^2}dx = \frac{1}{2}\int_{1}^{e^2} \frac{t^2+81}{t(t+9)^2}dt = \frac{1}{2}\int_{1}^{e^2}\left(\frac{1}{t} - \frac{18}{(t+9)^2}\right)dt$$

$$= \left[\frac{1}{2}\ln|t| + \frac{9}{t+9}\right]_1^{e^2} = \frac{e^2+99}{10(e^2+9)} \blacksquare$$

17. sol 1) $\int_{0}^{1} x^e(\ln x)^{2022}dx = \left[\frac{1}{e+1}x^{e+1}(\ln x)^{2022}\right]_0^1 - \frac{2022}{e+1}\int_0^1 x^e(\ln x)^{2021}dx$

$$= -\frac{2022}{e+1}\int_0^1 x^e(\ln x)^{2021}dx = -\frac{2022}{e+1}\left[\frac{1}{e+1}x^{e+1}(\ln x)^{2021}\right]_0^1 + \frac{2022\times 2021}{(e+1)^2}x^e(\ln x)^{2020}dx$$

$$= \cdots = \frac{2022!}{(e+1)^{2023}} \blacksquare$$

풀이의 엄밀성을 위해 음이 아닌 정수 n에 대하여 $\int_0^1 x^e(\ln x)^n dx = (-1)^n\frac{n!}{(e+1)^{n+1}}$ 임을 보이자. $n=0$일 때 $\int_0^1 x^e dx = \frac{1}{e+1} = (-1)^0\frac{0!}{(e+1)^1}$이므로 주어진 명제가 성립하고, 음이 아닌 정수 k에 대하여 $\int_0^1 x^e(\ln x)^k dx = (-1)^k\frac{k!}{(e+1)^{k+1}}$이 성립한다고 가정하자. 이때 $\int_0^1 x^e(\ln x)^{k+1}dx = \left[\frac{1}{e+1}x^{e+1}(\ln x)^{k+1}\right]_0^1 - \frac{k+1}{e+1}\int_0^1 x^e(\ln x)^k dx$ $= 0 - \frac{k+1}{e+1}\times(-1)^k\frac{k!}{(e+1)^{k+1}} = (-1)^{k+1}\frac{(k+1)!}{(e+1)^{k+2}}$이다. 따라서 수학적 귀납법에 의해 모든 음이 아닌 정수 n에 대하여 $\int_0^1 x^e(\ln x)^n dx = (-1)^n\frac{n!}{(e+1)^{n+1}}$이 성립한다.

sol 2) $\alpha > -1$, $\int_0^1 x^\alpha dx = \frac{1}{\alpha+1}$ 이므로 식의 양변을 α로 n번 미분하면

$$\frac{d^n}{d\alpha^n}\int_0^1 x^\alpha dx = \int_0^1 \frac{\partial^n}{\partial\alpha^n} x^\alpha dx = \int_0^1 x^\alpha (\ln x)^n dx = (-1)^n \frac{n!}{(\alpha+1)^{n+1}},\ n \in \mathbb{N} \cup \{0\}$$ 이다.

$$\therefore \int_0^1 x^e (\ln x)^{2022} dx = \frac{2022!}{(e+1)^{2023}}\ ■$$

18. sol 1) $t = \tan x$, $dt = \sec^2 x dx = (t^2+1)dx$

$$\int_0^{\frac{\pi}{4}} \frac{1}{\sin 2x + \cos 2x + 1} dx = \int_0^1 \frac{1}{\frac{2t}{1+t^2} + \frac{2}{1+t^2}} \times \frac{1}{1+t^2} dt = \int_0^1 \frac{1}{2t+2} dt$$

$$= \left[\frac{1}{2}\ln|t+1|\right]_0^1 = \frac{1}{2}\ln 2\ ■$$

sol 2) $\int_0^{\frac{\pi}{4}} \frac{1}{\sin 2x + \cos 2x + 1} dx = \int_0^{\frac{\pi}{4}} \frac{1}{2\sin x \cos x + 2\cos^2 x} dx$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{1}{(\sin x + \cos x)\cos x} dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} \left(\frac{\cos x - \sin x}{\sin x + \cos x} + \frac{\sin x}{\cos x}\right) dx$$

$$= \left[\frac{1}{2}\ln|\sin x + \cos x| - \ln|\cos x|\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\ln 2\ ■$$

19. $u = \frac{1}{x^3}$, $du = -\frac{3}{x^4}dx$

$$\int_{\sqrt[3]{3}}^4 \sqrt{\frac{x^3-3}{x^{11}}} dx = \frac{1}{3}\int_{\frac{1}{64}}^{\frac{1}{3}} \sqrt{1 - \frac{3}{x^3}} du = \frac{1}{3}\int_{\frac{1}{64}}^{\frac{1}{3}} \sqrt{1-3u}\, du = \left[-\frac{2}{27}(1-3u)^{\frac{3}{2}}\right]_{\frac{1}{64}}^{\frac{1}{3}}$$

$$= \frac{61\sqrt{61}}{6912}\ ■$$

20. $x = \sin t$, $dx = \cos t dt = \sqrt{1-x^2}\, dt$

$$\frac{2^{21}}{\pi}\int_0^1 \frac{x^{20}}{\sqrt{1-x^2}}dx = \frac{2^{21}}{\pi}\int_0^{\frac{\pi}{2}} \sin^{20}t dt = \frac{2^{21}}{\pi} \times \frac{19\times 17\times \cdots \times 1}{20\times 18\times \cdots \times 2} \times \frac{\pi}{2}$$

$$(\because \#10,\ \int_0^{\frac{\pi}{2}} \sin^n x dx = \frac{(n-1)!!}{n!!}\cdot\left(\frac{\pi}{2}\right)^{\frac{1+(-1)^n}{2}}\quad (n>0))$$

$$= 2^{20}\times\frac{19\times 17\times\cdots 1}{2^{10}\times 10!} = \frac{2^{10}\times\frac{20!}{2^{10}\times 10!}}{10!} = \frac{20!}{(10!)^2} = {}_{20}C_{10}\ \blacksquare$$

21. $I = \int_0^{\pi}\frac{x\tan x}{\sec x+\cos x}dx = \int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx = \int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}dx\ \ (x \mapsto \pi - x)$

$$= \int_0^{\pi}\frac{\pi\sin x}{1+\cos^2 x}dx - I = \frac{\pi}{2}\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$$

$$= -\frac{\pi}{2}\int_{\frac{\pi}{4}}^{-\frac{\pi}{4}}du = \frac{\pi^2}{4}\ \blacksquare\ (\cos x = \tan u,\ \sin x dx = -\sec^2 u du = -(1+\cos^2 x)du)$$

22. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin x\ln(1+e^x)dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin(-x)\ln(1+e^{-x})dx\ \ (x \mapsto -x)$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin x\ln\left(\frac{e^x}{1+e^x}\right)dx = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin x\left[\ln(1+e^x)+\ln\left(\frac{e^x}{1+e^x}\right)\right]dx$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}x\sin x dx = \frac{1}{2}[-x\cos x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x dx = [\sin x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 1\ \blacksquare$$

23. $\int_0^1\frac{1-x^{99}}{(1+x)(1+x^{100})}dx = \int_0^1\frac{(1+x^{100})-x^{99}(1+x)}{(1+x)(1+x^{100})}dx = \int_0^1\left(\frac{1}{1+x}-\frac{x^{99}}{1+x^{100}}\right)dx$

$$= \left[\ln|1+x| - \frac{1}{100}\ln|1+x^{100}|\right]_0^1 = \frac{99}{100}\ln 2\ \blacksquare$$

24. $\sin^6 x+\cos^6 x = (\sin^2 x+\cos^2 x)^3 - 3\sin^2 x\cos^2 x(\sin^2 x+\cos^2 x) = 1-3\sin^2 x\cos^2 x$

$= 1-\frac{3}{4}\sin^2 2x = \frac{1+3\cos^2 2x}{4}$ 이므로

$$\int_0^{\frac{\pi}{8}} \frac{\tan 2x}{\sqrt{\sin^6 x + \cos^6 x}}dx = 2\int_0^{\frac{\pi}{8}} \frac{\sin 2x}{\cos 2x\sqrt{1+3\cos^2 2x}}dx$$

$$= \int_{\frac{1}{\sqrt{2}}}^{1} \frac{1}{t\sqrt{1+3t^2}}dt \ (t = \cos 2x,\ dt = -2\sin 2x dx)$$

$$= \int_{\alpha}^{\frac{\pi}{3}} \frac{1}{\sin u}du \ (t = \frac{1}{\sqrt{3}}\tan u,\ dt = \frac{1}{\sqrt{3}}\sec^2 u du = \frac{1+3t^2}{\sqrt{3}}du,\ \tan\alpha = \sqrt{\frac{3}{2}})$$

$$= \int_{\alpha}^{\frac{\pi}{3}} \frac{\csc u(\csc u + \cot u)}{\csc u + \cot u}du = -\int_{\alpha}^{\frac{\pi}{3}} \frac{(\csc u + \cot u)'}{\csc u + \cot u}du = -[\ln|\csc u + \cot u|]_{\alpha}^{\frac{\pi}{3}}$$

$$= -\left[\ln\left|\frac{1+\cos u}{\sin u}\right|\right]_{\alpha}^{\frac{\pi}{3}} = \ln\left(\frac{\sqrt{5}+\sqrt{2}}{3}\right) \blacksquare$$

25. $$\int_0^{\frac{\pi}{4}} \frac{e^x \tan x(\cos 2x - 3)}{1+\cos 2x}dx = \int_0^{\frac{\pi}{4}} e^x \tan x \times \frac{2\cos^2 x - 4}{2\cos^2 x}dx = \int_0^{\frac{\pi}{4}} e^x \tan x(1-2\sec^2 x)dx$$

$$= \int_0^{\frac{\pi}{4}} e^x \tan x dx - \int_0^{\frac{\pi}{4}} e^x \cdot 2\tan x \sec^2 x dx = \int_0^{\frac{\pi}{4}} e^x \tan x dx - \int_0^{\frac{\pi}{4}} e^x \cdot (\sec^2 x)' dx$$

$$= [e^x \tan x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} e^x \sec^2 x dx - [e^x \sec^2 x]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} e^x \sec^2 x dx$$

$$= [e^x(\tan x - \sec^2 x)]_0^{\frac{\pi}{4}} = 1 - e^{\frac{\pi}{4}} \blacksquare$$

26. $u = x^{2021}$, $du = 2021x^{2020}dx$

$$\int_2^4 \frac{1}{x^{2022}-x}dx = \int_2^4 \frac{x^{2020}}{x^{4022}-x^{2021}}dx = \frac{1}{2021}\int_{2^{2021}}^{4^{2021}} \frac{1}{u^2-u}du = \frac{1}{2021}\int_{2^{2021}}^{2^{4042}} \left(\frac{1}{u-1} - \frac{1}{u}\right)du$$

$$= \frac{1}{2021}\left[\ln\left|\frac{u-1}{u}\right|\right]_{2^{2021}}^{2^{4042}} = \frac{1}{2021}\ln\left(\frac{2^{4042}-1}{2^{4042}} \times \frac{2^{2021}}{2^{2021}-1}\right) = \frac{1}{2021}\ln\left(\frac{2^{2021}+1}{2^{2021}}\right) \blacksquare$$

27. $u = x^{\sqrt{e}}$, $du = \sqrt{e}\,x^{\sqrt{e}-1}dx$

$$\int_3^9 \frac{1}{x(x^{\sqrt{e}}-1)}dx = \int_3^9 \frac{1}{x^{\sqrt{e}+1}-x}dx = \int_3^9 \frac{x^{\sqrt{e}-1}}{x^{2\sqrt{e}}-x^{\sqrt{e}}}dx = \frac{1}{\sqrt{e}}\int_{3^{\sqrt{e}}}^{9^{\sqrt{e}}} \frac{1}{u^2-u}du$$

$$= \frac{1}{\sqrt{e}}\int_{3^{\sqrt{e}}}^{3^{2\sqrt{e}}}\left(\frac{1}{u-1}-\frac{1}{u}\right)du = \frac{1}{\sqrt{e}}\left[\ln\left|\frac{u-1}{u}\right|\right]_{3^{\sqrt{e}}}^{3^{2\sqrt{e}}} = \frac{1}{\sqrt{e}}\ln\left(\frac{3^{2\sqrt{e}}-1}{3^{2\sqrt{e}}}\times\frac{3^{\sqrt{e}}}{3^{\sqrt{e}}-1}\right)$$

$$= \frac{1}{\sqrt{e}}\ln\left(\frac{3^{\sqrt{e}}+1}{3^{\sqrt{e}}}\right) \blacksquare$$

28. $$\int_0^{\frac{\pi}{2}} \sin 2022x \cdot \sin^{2020}x dx = \int_0^{\frac{\pi}{2}} \sin(2021x+x)\cdot \sin^{2020}x dx$$

$$= \int_0^{\frac{\pi}{2}} \sin 2021x \cdot \cos x \cdot \sin^{2020}x dx + \int_0^{\frac{\pi}{2}} \cos 2021x \cdot \sin^{2021}x dx$$

$$= \sin 2021x \cdot \frac{1}{2021}\sin^{2021}x - \frac{2021}{2021}\int_0^{\frac{\pi}{2}} \cos 2021x \cdot \sin^{2021}x dx + \int_0^{\frac{\pi}{2}} \cos 2021x \cdot \sin^{2021}x dx$$

$$= \left[\frac{1}{2021}\sin 2021x \cdot \sin^{2021}x dx\right]_0^{\frac{\pi}{2}} = \frac{1}{2021} \blacksquare$$

29. $t = x^2 + \frac{1}{x^2} - 1$, $dt = 2\left(x - \frac{1}{x^3}\right)dx$

$$\int_1^{\sqrt{2}} \frac{x^4-1}{x^2\sqrt{x^4-x^2+1}}dx = \int_1^{\sqrt{2}} \frac{x^4-1}{x^3\sqrt{x^2+\frac{1}{x^2}-1}}dx = \int_0^{\sqrt{2}} \frac{x-1/x^3}{\sqrt{x^2+\frac{1}{x^2}-1}}dx$$

$$= \frac{1}{2}\int_1^{\frac{3}{2}} \frac{1}{\sqrt{t}}dt = [\sqrt{t}\,]_1^{\frac{3}{2}} = \frac{\sqrt{6}}{2} - 1 \blacksquare$$

30. $$\int_3^4 \frac{x+4}{x^3+3x^2-10x}dx = \int_3^4 \frac{x+4}{x(x+5)(x-2)}dx = \int_3^4\left(-\frac{2}{5x} - \frac{1}{35(x+5)} + \frac{3}{7(x-2)}\right)dx$$

$$= \left[-\frac{2}{5}\ln|x| - \frac{1}{35}\ln|x+5| + \frac{3}{7}\ln|x-2|\right]_3^4 = \frac{12}{35}\ln 3 - \frac{10}{35}\ln 2 = \frac{2}{35}\ln\left(\frac{729}{32}\right) \blacksquare$$

31. sol 1) $t=\sec x+\tan x,\ dt=\sec x(\sec x+\tan x)dx$

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\frac{\sec^2 x}{(\sec x+\tan x)^{3/2}}dx=\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\frac{\sec x(\sec x+\tan x)+\sec x(\sec x-\tan x)}{(\sec x+\tan x)^{3/2}}dx$$

$$=\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\frac{\sec x(\sec x+\tan x)}{(\sec x+\tan x)^{3/2}}dx+\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\frac{\sec x(\sec x-\tan x)}{(\sec x+\tan x)^{3/2}}dx$$

$$=\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\frac{\sec x(\sec x+\tan x)}{(\sec x+\tan x)^{3/2}}dx+\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\frac{\sec x(\sec x+\tan x)}{(\sec x+\tan x)^{7/2}}dx$$

$$=\frac{1}{2}\int_{2-\sqrt{3}}^{2+\sqrt{3}}t^{-3/2}dt+\frac{1}{2}\int_{2-\sqrt{3}}^{2+\sqrt{3}}t^{-7/2}dt=\left[-t^{-1/2}-\frac{1}{5}t^{-5/2}\right]_{2-\sqrt{3}}^{2+\sqrt{3}}=\frac{24}{5}\sqrt{2}\ ■$$

sol 2) $t=\tan x,\ dt=\sec^2 x dx,\ u=t+\sqrt{1+t^2},\ t=\frac{1}{2}\left(u-\frac{1}{u}\right),\ dt=\frac{1}{2}\left(1+\frac{1}{u^2}\right)du$

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\frac{\sec^2 x}{(\sec x+\tan x)^{3/2}}dx=\int_{-\sqrt{3}}^{\sqrt{3}}\frac{1}{(t+\sqrt{1+t^2})^{3/2}}dt=\frac{1}{2}\int_{2-\sqrt{3}}^{2+\sqrt{3}}(u^{-3/2}+u^{-7/2})du$$

$$=\left[-u^{-1/2}-\frac{1}{5}u^{-5/2}\right]_{2-\sqrt{3}}^{2+\sqrt{3}}=\frac{24}{5}\sqrt{2}\ ■$$

32. $I=\int_0^{\frac{\pi}{2}}\ln(\cos x)dx=\int_0^{\frac{\pi}{2}}\ln(\sin x)dx\ \left(x\mapsto\frac{\pi}{2}-x\right)$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(\sin x\cos x)dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln\left(\frac{1}{2}\sin 2x\right)dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(\sin 2x)dx-\frac{\pi}{4}\ln 2$$

$$=\frac{1}{4}\int_0^{\pi}\ln(\sin t)dt-\frac{\pi}{4}\ln 2\ (t=2x,\ dt=2dx)$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(\sin t)dt-\frac{\pi}{4}\ln 2=\frac{1}{2}I-\frac{\pi}{4}\ln 2,\quad \frac{1}{2}I=-\frac{\pi}{4}\ln 2$$

$$\therefore I=\int_0^{\frac{\pi}{2}}\ln(\cos x)dx=-\frac{\pi}{2}\ln 2\ ■$$

33. $\int_{1}^{\sqrt[6]{6}} \frac{6}{x(x^6+2)^2}dx = \frac{3}{2}\int_{1}^{\sqrt[6]{6}}\left(\frac{1}{x} - \frac{x^{11}+4x^5}{(x^6+2)^2}\right)dx = \frac{3}{2}\int_{1}^{\sqrt[6]{6}}\left(\frac{1}{x} - \frac{x^5(x^6+2)+2x^5}{(x^6+2)^2}\right)dx$

$= \frac{3}{2}\int_{1}^{\sqrt[6]{6}}\left(\frac{1}{x} - \frac{x^5}{x^6+2} - \frac{2x^5}{(x^6+2)^2}\right)dx = \frac{3}{2}\left[\ln|x| - \frac{1}{6}\ln|x^6+2| + \frac{1}{3(x^6+2)}\right]_1^{\sqrt[6]{6}}$

$= \frac{1}{2}\ln\left(\frac{3}{2}\right) - \frac{5}{48}$ ■

34. $x = \sqrt{1-t^2},\ dx = -\frac{t}{\sqrt{1-t^2}}dt$

$I = \int_0^1 \frac{x^2}{x+\sqrt{1-x^2}}dx = \int_1^0 \frac{1-t^2}{t+\sqrt{1-t^2}} \times \left(-\frac{t}{\sqrt{1-t^2}}dt\right) = \int_0^1 \frac{t\sqrt{1-t^2}}{t+\sqrt{1-t^2}}dt$

$= \frac{1}{2}\int_0^1\left[\frac{x^2}{x+\sqrt{1-x^2}} + \frac{x\sqrt{1-x^2}}{x+\sqrt{1-x^2}}\right]dx = \frac{1}{2}\int_0^1 x\,dx = \frac{1}{2}\left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{4}$ ■

35. $\int_0^{\frac{\pi}{4}} \frac{1}{1-3\cos^2 x}dx = \int_0^{\frac{\pi}{4}} \sec^2 x \cdot \frac{1}{\tan^2 x - 2}dx = \left[\frac{1}{2\sqrt{2}}\ln\left|\frac{\tan x - \sqrt{2}}{\tan x + \sqrt{2}}\right|\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2\sqrt{2}}\ln(3-2\sqrt{2}) = \frac{1}{\sqrt{2}}\ln(\sqrt{2}-1)$ ■

36. $\int_0^{\frac{1}{2}} \frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}}dx = \int_0^{\frac{1}{2}}\left(\frac{e^x}{(1-x)\sqrt{1-x^2}} + e^x\sqrt{\frac{1+x}{1-x}}\right)dx$

$= \int_0^{\frac{1}{2}}\left(e^x\left(\sqrt{\frac{1+x}{1-x}}\right)' + (e^x)'\sqrt{\frac{1+x}{1-x}}\right)dx = \left[e^x\sqrt{\frac{1+x}{1-x}}\right]_0^{\frac{1}{2}} = \sqrt{3e}-1$ ■

37. $\int_1^{\frac{\pi}{2}} x^{\sin x - 1}(x\cos x \ln x + \sin x)dx$

고등학교 과정으로 위와 같은 적분을 일반적으로 계산하는 것은 어렵다. $x^{\sin x}$ 항이 문제이므로 적분의 결과에 $x^{\sin x}$가 있을 것이라 예상할 수 있고, $y = x^{\sin x}$라 하면 $\ln y = \sin x \ln x$, $\frac{y'}{y} = \cos x \ln x + \frac{\sin x}{x}$이므로 $y' = y\left(\cos x \ln x + \frac{\sin x}{x}\right)$이고 이는 문제의 피적분함수와 같다.

$$\therefore \int_1^{\frac{\pi}{2}} x^{\sin x - 1}(x\cos x \ln x + \sin x)dx = \left[x^{\sin x}\right]_1^{\frac{\pi}{2}} = \frac{\pi}{2} - 1 \ ■$$

38. $y = \sqrt[3]{1-x^7}$ 의 역함수는 $y = \sqrt[7]{1-x^3}$ 이므로 $f(x) = \sqrt[3]{1-x^7}$ 이라 하면

$$\int_0^1 \left(\sqrt[3]{1-x^7} - \sqrt[7]{1-x^3}\right)dx = \int_0^1 f(x)dx - \int_0^1 f^{-1}(x)dx$$

$$= \int_0^1 f(x)dx - \int_1^0 xf'(x)dx \ \ (x \mapsto f(x))$$

$$= \int_0^1 [f(x) + xf'(x)]dx = [xf(x)]_0^1 = \left[x\sqrt[3]{1-x^7}\right]_0^1 = 0 \ ■$$

39. $f(x) = \sqrt{1+x^3}$ 이라 하면 $f^{-1}(x) = \sqrt[3]{x^2-1}$ 이고 $\sqrt[3]{x^2+2x} = f^{-1}(x+1)$ 이므로

$$\int_0^2 \left(\sqrt{1+x^3} + \sqrt[3]{x^2+2x}\right)dx = \int_0^2 (f(x) - f^{-1}(x+1))dx$$

$$= \int_0^2 f(x)dx - \int_1^3 f^{-1}(u)du \ \ (u = x+1,\ du = dx)$$

$$= \int_0^2 f(x)dx - \int_{f^{-1}(1)}^{f^{-1}(3)} tf'(t)dt = \int_0^2 f(x)dx - \int_0^2 tf'(t)dt \ \ (u = f(t),\ du = f'(t)dt)$$

$$= \int_0^2 (f(x) - xf'(x))dx = [xf(x)]_0^2 = 2f(2) = 6 \ ■$$

40. $$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sqrt{\sin 2x}}dx = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right)}{1+\sqrt{\sin(\pi - 2x)}}dx = \int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sqrt{\sin 2x}}dx \quad \left(x \mapsto \frac{\pi}{2} - x\right)$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{1+\sqrt{\sin 2x}}dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{1+\sqrt{2\sin x\cos x}}dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{(\sin x - \cos x)'}{1+\sqrt{1-(\sin x - \cos x)^2}}dx$$

$$= \frac{1}{2}\int_{-1}^1 \frac{1}{1+\sqrt{1-u^2}}du \ \ (u = \sin x - \cos x,\ du = (\sin x - \cos x)'dx)$$

$$= \int_0^1 \frac{1}{1+\sqrt{1-u^2}}du = \int_0^{\frac{\pi}{2}} \frac{\cos t}{1+\cos t}dt \ \ (u = \sin t,\ du = \cos t dt)$$

$$= \int_0^{\frac{\pi}{2}} \left(1 - \frac{1}{1+\cos t}\right) dt = \int_0^{\frac{\pi}{2}} \left(1 - \frac{1}{2}\sec^2\frac{t}{2}\right) dt = \left[t - \tan\frac{t}{2}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1 \ \blacksquare$$

41. sol 1) $t = -\ln x,\ x = e^{-t},\ dx = -e^{-t}dt$

$$\int_0^1 (\ln x)^{2022} dx = \int_0^{\infty} t^{2022} e^{-t} dt = \Gamma(2023) = 2022! \ \blacksquare$$

sol 2) $J_n := \int_0^1 (-\ln x)^n dx = [x \cdot (-\ln x)^n]_0^1 - \int_0^1 x \cdot n(-\ln x)^{n-1} \cdot \frac{-1}{x} dx$

$$= 0 - \lim_{x \to 0} [x \cdot (-\ln x)^n] + n \int_0^1 (-\ln x)^{n-1} dx = nJ(n-1)$$

($\because x = e^{-t},\ \lim_{x \to 0} [x \cdot (-\ln x)^n] = \lim_{t \to \infty} \frac{t^n}{e^t} = 0$ by l' Hospital' s rule)

$$J_n = nJ_{n-1} = n(n-1)J_{n-2} = \cdots = n(n-1)\cdots 1 \cdot J_0 = n!$$

$$\therefore \int_0^1 (\ln x)^{2022} dx = J_{2022} = 2022! \ \blacksquare$$

42. $n \in \mathbb{N} \cup \{0\},\ a_n := \int_0^{\frac{\pi}{2}} \cos^n x \cos nx dx$

$$a_n = \int_0^{\frac{\pi}{2}} \cos^n x [\cos(n+1)x \cos x + \sin(n+1)x \sin x] dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n+1} x \cos(n+1)x dx + \int_0^{\frac{\pi}{2}} \cos^n x \sin(n+1)x \sin x dx$$

$$= a_{n+1} + \left[\sin(n+1)x \cdot \frac{-1}{n+1} \cos^{n+1} x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \cos(n+1)x(-\cos^{n+1} x) dx$$

$$= 2a_{n+1}$$

$$a_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2},\ a_1 = \int_0^{\frac{\pi}{2}} \cos^2 dx = \left[\frac{1}{2}x + \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

$$a_n = \int_0^{\frac{\pi}{2}} \cos^n x \cos nx dx = \pi \cdot \left(\frac{1}{2}\right)^{n+1}$$

$$\therefore a_{999} = \int_0^{\frac{\pi}{2}} \cos^{999} x \cos 999x dx = \frac{\pi}{2^{1000}} \ \blacksquare$$

43. sol 1) $I = \int_0^{\frac{\pi}{2}} \frac{x}{\tan x} dx = [x \ln(\sin x)]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \ln(\sin x) dx = -\int_0^{\frac{\pi}{2}} \ln(\sin x) dx$

$$= -\int_0^{\frac{\pi}{2}} \ln(\cos x) dx \ \left(x \mapsto \frac{\pi}{2} - x\right)$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\sin x \cos x) dx = -\frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{2}\sin 2x\right) dx = -\frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\sin 2x) dx + \frac{\pi}{4}\ln 2$$

$$= -\frac{1}{4}\int_0^{\pi} \ln(\sin t) dt + \frac{\pi}{4}\ln 2 \ (t = 2x,\ dt = 2dx)$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\sin t) dt + \frac{\pi}{4}\ln 2 = \frac{1}{2}I + \frac{\pi}{4}\ln 2, \quad \frac{1}{2}I = \frac{\pi}{4}\ln 2$$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{x}{\tan x} dx = \frac{\pi}{2}\ln 2 \ \blacksquare$$

sol 2) 파인만 테크닉 : $I(\alpha) := \int_0^{\frac{\pi}{2}} \frac{\tan^{-1}(\alpha \tan x)}{\tan x} dx$

$$\frac{d}{d\alpha} I(\alpha) = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial \alpha} \frac{\tan^{-1}(\alpha \tan x)}{\tan x} dx = \int_0^{\frac{\pi}{2}} \frac{1}{1+\alpha^2 \tan^2 x} dx$$

$$= \int_0^{\infty} \frac{1}{(1+\alpha^2 t^2)(1+t^2)} dt \ (t = \tan x,\ dt = \sec^2 x dx)$$

$$= \frac{1}{\alpha^2 - 1}\int_0^{\infty} \left(\frac{\alpha^2}{1+\alpha^2 t^2} - \frac{1}{1+t^2}\right) dt = \frac{1}{\alpha^2 - 1}\left[\alpha \tan^{-1}(\alpha t) - \tan^{-1} t\right]_0^{\infty} = \frac{\pi}{2(\alpha+1)}$$

$I(\alpha) = \frac{\pi}{2}\ln|\alpha+1| + C, \quad C = I(0) = 0$에서 $I(\alpha) = \frac{\pi}{2}\ln|\alpha+1|$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{x}{\tan x} dx = I(1) = \frac{\pi}{2}\ln 2 \ \blacksquare$$

44. $I(\alpha) := \int_0^{\alpha} \sin^2\left(x - \sqrt{\alpha^2 - x^2}\right) dx = \int_{\frac{\pi}{2}}^{0} \sin^2(\alpha\cos t - \alpha\sin t) \cdot (-\alpha\sin t dt)$

$$= \int_0^{\frac{\pi}{2}} \sin^2(\alpha\cos t - \alpha\sin t)\alpha\sin t dt \ \ (x = \alpha\cos t,\ dx = -\alpha\sin t dt)$$

$$I(\alpha) := \int_0^{\alpha} \sin^2\left(x - \sqrt{\alpha^2 - x^2}\right) dx = \int_0^{\frac{\pi}{2}} \sin^2(\alpha\sin t - \alpha\cos t)\alpha\cos t dt$$

$$= \int_0^{\frac{\pi}{2}} \sin^2(\alpha\cos t - \alpha\sin t)\alpha\cos t dt \ \ (x = \alpha\sin t,\ dx = \alpha\cos t dt)$$

$$I(\alpha) = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2(\alpha(\sin t - \cos t))\alpha(\sin t + \cos t) dt$$

$$= \frac{1}{2}\int_{-\alpha}^{\alpha} \sin^2 u du \ \ (u = \alpha(\sin t - \cos t),\ du = \alpha(\cos t + \sin t)dt)$$

$$= \frac{1}{4}\int_{-\alpha}^{\alpha} (1 - \cos 2u) du = \frac{1}{4}\left[u - \frac{1}{2}\sin 2u\right]_{-\alpha}^{\alpha} = \frac{\alpha}{2} - \frac{1}{4}\sin 2\alpha$$

$$\therefore \int_0^{\frac{\pi}{12}} \sin^2\left(x - \sqrt{\frac{\pi^2}{12^2} - x^2}\right) dx = I\left(\frac{\pi}{12}\right) = \frac{\pi}{24} - \frac{1}{8} \ \blacksquare$$

45. $\int_1^{e^2} \frac{\ln(xe^{x+1})}{(x+1)^2 + [\ln(x^x)]^2} dx = \int_1^{e^2} \frac{x+1+\ln x}{(x+1)^2 + (x\ln x)^2} dx = \int_1^{e^2} \frac{\frac{x+1+\ln x}{(x+1)^2}}{1 + \left(\frac{x\ln x}{x+1}\right)^2} dx$

$$= \int_1^{e^2} \frac{\left(\frac{x\ln x}{x+1}\right)'}{1 + \left(\frac{x\ln x}{x+1}\right)^2} dx \ \ \left(\tan\theta = \frac{x\ln x}{x+1},\ \sec^2\theta d\theta = \left(\frac{x\ln x}{x+1}\right)' dx\right)$$

$$= \int_0^{\alpha} \frac{1}{1 + \tan^2\theta} \cdot \sec^2\theta d\theta = \int_0^{\alpha} d\theta \ \ \left(\tan\alpha = \frac{2e^2}{e^2+1},\ \alpha = \tan^{-1}\left(\frac{2e^2}{e^2+1}\right)\right)$$

$$= [\theta]_0^{\alpha} = \left[\tan^{-1}\left(\frac{x\ln x}{x+1}\right)\right]_1^{e^2} = \tan^{-1}\left(\frac{2e^2}{e^2+1}\right) \ ■$$

46. $f(x) = \sqrt{x+\ln x}$, $f'(x) = \dfrac{1+1/x}{2\sqrt{x+\ln x}} = \dfrac{x+1}{2xf(x)}$

$$\int_1^e \frac{(x+1)^2+(3x+1)\sqrt{x+\ln x}}{2x\sqrt{x+\ln x}\,(x+\sqrt{x+\ln x})}dx = \int_1^e \frac{(x+1)^2+(3x+1)f(x)}{2xf(x)(x+f(x))}dx$$

$$= \int_1^e\left(\frac{(x+1)^2}{2xf(x)(x+f(x))} + \frac{(x+1)f(x)}{2xf(x)(x+f(x))} + \frac{2xf(x)}{2xf(x)(x+f(x))}\right)dx$$

$$= \int_1^e\left(\frac{(x+1)f'(x)}{x+f(x)} + \frac{f(x)f'(x)}{x+f(x)} + \frac{1}{x+f(x)}\right)dx$$

$$= \int_1^e\left(\frac{xf'(x)+f(x)f'(x)+f'(x)}{x+f(x)} + \frac{1}{x+f(x)}\right)dx$$

$$= \int_1^e\left(f'(x) + \frac{1+f'(x)}{x+f(x)}\right)dx = [f(x)+\ln|x+f(x)|]_1^e$$

$$= [\sqrt{x+\ln x} + \ln|x+\sqrt{x+\ln x}|]_1^e = \sqrt{1+e}+\ln(e+\sqrt{1+e})-\ln 2-1 \ ■$$

47. sol 1) $\cos 2x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$, $t^2 = 1-\tan^2 x$, $2tdt = -2\tan x\sec^2 x dx$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sqrt{\cos 2x}}{\sin x}dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sqrt{1-\tan^2 x}}{\tan x}dx = \int_{\sqrt{\frac{2}{3}}}^{0} \frac{t}{\tan x}\cdot\frac{-t}{\tan x\sec^2 x}dt = \int_0^{\sqrt{\frac{2}{3}}} \frac{t^2}{(1-t^2)(2-t^2)}dt$$

$$= \int_0^{\sqrt{\frac{2}{3}}}\left(\frac{1}{1-t^2} - \frac{2}{2-t^2}\right)dt = \left[\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| - \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}+t}{\sqrt{2}-t}\right|\right]_0^{\sqrt{\frac{2}{3}}}$$

$$= \frac{1}{2}\ln(5+2\sqrt{6}) - \frac{1}{\sqrt{2}}\ln(2+\sqrt{3}) \ ■$$

sol 2) $u = \dfrac{\cos x}{\sqrt{\cos 2x}}$, $du = \dfrac{-\sin x\sqrt{\cos 2x} + \cos x\cdot\dfrac{\sin 2x}{\sqrt{\cos 2x}}}{\cos 2x}dx$

$$= \frac{-\sin x\cos 2x + \cos x \sin 2x}{(\sqrt{\cos 2x})^3}dx = \frac{\sin x}{(\sqrt{\cos 2x})^3}dx$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sqrt{\cos 2x}}{\sin x}dx = \int_{\sqrt{\frac{3}{2}}}^{\infty} \frac{\cos^2 2x}{\sin^2 x}du = \int_{\sqrt{\frac{3}{2}}}^{\infty} \frac{\cos 2x}{\cos^2 x - \cos^2 x + \sin^2 x} \cdot \frac{\cos 2x}{2\cos^2 x - \cos^2 x + \sin^2 x}du$$

$$= \int_{\sqrt{\frac{3}{2}}}^{\infty} \frac{\cos 2x}{\cos^2 x - \cos 2x} \cdot \frac{\cos 2x}{2\cos^2 x - \cos 2x}du = \int_{\sqrt{\frac{3}{2}}}^{\infty} \frac{1}{(u^2-1)(2u^2-1)}du$$

$$= \int_{\sqrt{\frac{3}{2}}}^{\infty} \left(\frac{1}{u^2-1} - \frac{2}{2u^2-1}\right)du = \left[\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| - \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}u-1}{\sqrt{2}u+1}\right|\right]_{\sqrt{\frac{3}{2}}}^{\infty}$$

$$= -\frac{1}{2}\ln(5-2\sqrt{6}) + \frac{1}{\sqrt{2}}\ln(2-\sqrt{3}) = \frac{1}{2}\ln(5+2\sqrt{6}) - \frac{1}{\sqrt{2}}\ln(2+\sqrt{3}) \ \blacksquare$$

48. $I(\alpha) := \int_0^{\pi} \ln(\alpha^2 - 2\alpha\cos x + 1)dx = \int_0^{\pi} \ln(\alpha^2 + 2\alpha\cos x + 1)dx \ (x \mapsto \pi - x)$이므로

$I(\alpha) = I(-\alpha)$이다. 또한

$$I(\alpha) + I(-\alpha) = \int_0^{\pi} \ln\left[(\alpha^2 - 2\alpha\cos x + 1)(\alpha^2 + 2\alpha\cos x + 1)\right]dx$$

$$= \int_0^{\pi} \ln\left[(1+\alpha^2)^2 - (2\alpha\cos x)^2\right]dx = \int_0^{\pi} \ln\left[1 + 2\alpha^2 + \alpha^4 - 2\alpha^2(1+\cos 2x)\right]dx$$

$$= \int_0^{\pi} \ln\left[\alpha^4 - 2\alpha^2\cos 2x + 1\right]dx = \frac{1}{2}\int_0^{2\pi} \ln\left[\alpha^4 - 2\alpha^2\cos x + 1\right]dx \ \left(x \mapsto \frac{1}{2}x\right)$$

$$= \frac{1}{2}I(\alpha^2) + \frac{1}{2}\int_{\pi}^{2\pi} \ln\left[\alpha^4 - 2\alpha^2\cos x + 1\right]dx$$

$$= \frac{1}{2}I(\alpha^2) + \frac{1}{2}\int_0^{\pi} \ln\left[\alpha^4 - 2\alpha^2\cos x + 1\right]dx \ (x \mapsto 2\pi - x)$$

$$= I(\alpha^2)$$

이고, $I(\alpha) = I(-\alpha)$이므로 $I(\alpha) = \frac{1}{2}I(\alpha^2)$이 성립한다. 이때 $0 \le \alpha < 1$이면 이를 n번

반복하여 $I(\alpha) = \frac{1}{2^n}I(\alpha^{2^n})$을 얻고, n을 무한대로 보내면 우변은 0으로 수렴하므로

$I(\alpha) = 0$이다. $\left(I(\alpha) = \frac{1}{2}I(\alpha^2)\right.$에서 $I(0) = 0$, $\left.I(1) = 0\right)$

$\alpha > 1$인 경우 $0 < \frac{1}{\alpha} < 1$이므로 $I\left(\frac{1}{\alpha}\right) = 0$이고

$I(\alpha) = \int_0^\pi \ln(\alpha^2 - 2\alpha\cos x + 1)dx = \int_0^\pi \ln\left[\alpha^2\left(1 - \frac{2}{\alpha}\cos x + \frac{1}{\alpha^2}\right)\right]dx$

$= \int_0^\pi \ln\alpha^2 dx + \int_0^\pi \ln\left(1 - \frac{2}{\alpha}\cos x + \frac{1}{\alpha^2}\right)dx = \int_0^\pi 2\ln\alpha dx + I\left(\frac{1}{\alpha}\right) = 2\pi\ln\alpha$이다.

한편 $I(\alpha) = I(-\alpha)$이므로 최종적으로 $I(\alpha)$는 다음과 같다.

$$I(\alpha) = \begin{cases} 0 & (|\alpha| \le 1) \\ 2\pi\ln|\alpha| & (otherwise) \end{cases}$$

$\therefore \int_0^\pi \ln(\pi^2 - 2\pi\cos x + 1)dx = I(\pi) = 2\pi\ln\pi$ ■

※ $|\alpha| < 1$일 때 리만적분의 정의를 이용하여 $I(\alpha) = 0$을 유도할 수도 있다.

$x^{2n} = 1$은 복소 범위에서 $2n$개의 근 $e^{i\frac{k\pi}{n}} = \cos\left(\frac{k\pi}{n}\right) + i\sin\left(\frac{k\pi}{n}\right)$

$(k \in \mathbb{N},\ 0 \le k \le 2n-1)$를 가진다. $1 \le j \le n-1$에 대하여 j번째 근은

$x_j = \cos\left(\frac{j\pi}{n}\right) + i\sin\left(\frac{j\pi}{n}\right)$, $2n-j$번째 근은

$x_{2n-j} = \cos\left(\frac{(2n-j)\pi}{n}\right) + i\sin\left(\frac{(2n-j)\pi}{n}\right) = \cos\left(\frac{j\pi}{n}\right) - i\sin\left(\frac{j\pi}{n}\right)$이므로

$x_j + x_{2n-j} = 2\cos\left(\frac{j\pi}{n}\right)$, $x_j x_{2n-j} = 1$이고 x_j, x_{2n-j}를 두 근으로 가지는 이차방정식은 $x^2 - 2\cos\left(\frac{j\pi}{n}\right) + 1 = 0$이다. 또한 $j = n$, $j = 0$일 때는 $x_n = -1$, $x_0 = 1$이므로 $x^{2n} - 1$은 다음과 같이 인수분해할 수 있다.

$$x^{2n} - 1 = (x^2 - 1)\left(x^2 - 2\cos\left(\frac{\pi}{n}\right) + 1\right)\left(x^2 - 2\cos\left(\frac{2\pi}{n}\right) + 1\right) \cdots \left(x^2 - 2\cos\left(\frac{(n-1)\pi}{n}\right) + 1\right)$$

$$= (x^2 - 1)\prod_{k=1}^{n-1}\left(x^2 - 2x\cos\left(\frac{k\pi}{n}\right) + 1\right)$$

이때 $x = \alpha$를 대입하면 $\alpha^2 \ne 1$이므로 $\prod_{k=1}^{n-1}\left(\alpha^2 - 2\alpha\cos\left(\frac{k\pi}{n}\right) + 1\right) = \frac{\alpha^{2n} - 1}{\alpha^2 - 1}$이다.

한편 리만적분의 정의에 의해

$$I(\alpha) = \int_0^{\pi} \ln(\alpha^2 - 2\alpha\cos x + 1)dx = \lim_{n\to\infty} \frac{\pi}{n} \sum_{k=0}^{n-1} \ln\left(\alpha^2 - 2\alpha\cos\left(\frac{k\pi}{n}\right) + 1\right)$$

$$= \lim_{n\to\infty}\left[\frac{2\pi}{n}\ln(1-\alpha) + \frac{\pi}{n}\ln\left(\prod_{k=1}^{n-1}\left(\alpha^2 - 2\alpha\cos\left(\frac{k\pi}{n}\right)+1\right)\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{2\pi}{n}\ln(1-\alpha) + \frac{\pi}{n}\ln\left(\frac{1-\alpha^{2n}}{1-\alpha^2}\right)\right] = 0$$이다. ■

49. $r = \frac{\pi}{e} - \sqrt{\frac{\pi^2}{e^2} - 1} = \frac{\pi - \sqrt{\pi^2 - e^2}}{e}$이라 하면

$$\ln(e\cos x + \pi) = \ln\left(\frac{\pi + \sqrt{\pi^2 - e^2}}{2}\right) + \ln\left(\frac{2e\cos x + 2\pi}{\pi + \sqrt{\pi^2 - e^2}}\right)$$

$$= \ln\left(\frac{\pi + \sqrt{\pi^2 - e^2}}{2}\right) + \ln\left(\frac{2e\cos x}{\pi + \sqrt{\pi^2 - e^2}} + \frac{2\pi}{\pi + \sqrt{\pi^2 - e^2}}\right)$$

$$= \ln\left(\frac{\pi + \sqrt{\pi^2 - e^2}}{2}\right) + \ln\left(\frac{2(\pi - \sqrt{\pi^2 - e^2})}{e}\cos x + \frac{2\pi(\pi - \sqrt{\pi^2 - e^2})}{e^2}\right)$$

$$= \ln\left(\frac{\pi + \sqrt{\pi^2 - e^2}}{2}\right) + \ln\left(\frac{2(\pi - \sqrt{\pi^2 - e^2})}{e}\cos x + \frac{2\pi(\pi - \sqrt{\pi^2 - e^2})}{e^2}\right)$$

$$= \ln\left(\frac{\pi + \sqrt{\pi^2 - e^2}}{2}\right) + \ln\left(\frac{2(\pi - \sqrt{\pi^2 - e^2})}{e}\cos x + \frac{\pi^2 + \pi^2 - e^2 - 2\pi\sqrt{\pi^2 - e^2}}{e^2} + 1\right)$$

$$= \ln\left(\frac{\pi + \sqrt{\pi^2 - e^2}}{2}\right) + \ln(r^2 + 2r\cos x + 1)$$

한편 $x > 1$이면 $x - \sqrt{x^2 - 1} < 1$이므로 $\frac{\pi}{e} > 1$에서 $0 < r < 1$, $-1 < -r < 0$이고 48번 에 의해 $\int_0^{\pi} \ln(r^2 + 2r\cos x + 1)dx = 0$이다.

$$\therefore \int_0^{\pi} \ln(e\cos x + \pi)dx = \int_0^{\pi}\left[\ln\left(\frac{\pi + \sqrt{\pi^2 - e^2}}{2}\right) + \ln(r^2 + 2r\cos x + 1)\right]dx$$

$$= \int_0^{\pi} \ln\left(\frac{\pi + \sqrt{\pi^2 - e^2}}{2}\right) dx = \pi \ln\left(\frac{\pi + \sqrt{\pi^2 - e^2}}{2}\right) \blacksquare$$

50. 황금비 $\phi = \dfrac{1+\sqrt{5}}{2}$ 는 이차방정식 $\phi^2 - \phi - 1 = 0$의 한 근이다. 한편 분모에 $(1+x^{\phi})^{\phi}$이 있으므로 이를 만들기 위해 $(1+x^{\phi})^{1-\phi}$을 미분해보면 $\dfrac{d}{dx}(1+x^{\phi})^{1-\phi} = \dfrac{\phi(1-\phi)x^{\phi-1}}{(1+x^{\phi})^{\phi}} = -\dfrac{x^{\phi-1}}{(1+x^{\phi})^{\phi}}$ 이다. 따라서

$$\int_0^1 \frac{1}{(1+x^{\phi})^{\phi}} dx = \int_0^1 \frac{1+x^{\phi}-x^{\phi}}{(1+x^{\phi})^{\phi}} dx = \int_0^1 \left(\frac{1}{(1+x^{\phi})^{\phi-1}} - \frac{x^{\phi}}{(1+x^{\phi})^{\phi}}\right) dx$$

$$= \int_0^1 \left(x'(1+x^{\phi})^{1-\phi} + x\left[(1+x^{\phi})^{1-\phi}\right]'\right) dx = \left[x(1+x^{\phi})^{1-\phi}\right]_0^1 = 2^{1-\phi} = 2^{\frac{1-\sqrt{5}}{2}} \blacksquare$$

※ 위 결과와 이상적분의 정의를 이용하면 $\displaystyle\int_0^{\infty} \frac{1}{(1+x^{\phi})^{\phi}} dx = 1$이 성립함을 쉽게 보일 수 있으며, 황금비가 관여하는 아름다운 적분의 대표적인 예시이다.

$$\int_0^{\infty} \frac{1}{(1+x^{\phi})^{\phi}} dx = \left[x(1+x^{\phi})^{1-\phi}\right]_0^{\infty} = \lim_{x\to\infty} x(1+x^{\phi})^{1-\phi} - 0$$

$$= \lim_{x\to\infty} \frac{x}{(1+x^{\phi})^{\phi-1}} = 1 \blacksquare \ (\because \phi^2 - \phi = 1)$$