

정적분 100題

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(1) 각변환

$$\sin(-\theta) = -\sin\theta$$

$$\sin(\pi + \theta) = -\sin\theta$$

$$\sin(\pi - \theta) = \sin\theta$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

$$\sin\left(\frac{3}{2}\pi + \theta\right) = -\cos\theta$$

$$\sin\left(\frac{3}{2}\pi - \theta\right) = -\cos\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\cos(\pi + \theta) = -\cos\theta$$

$$\cos(\pi - \theta) = -\cos\theta$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\cos\left(\frac{3}{2}\pi + \theta\right) = \sin\theta$$

$$\cos\left(\frac{3}{2}\pi - \theta\right) = -\sin\theta$$

$$\tan(-\theta) = -\tan\theta$$

$$\tan(\pi + \theta) = \tan\theta$$

$$\tan(\pi - \theta) = -\tan\theta$$

$$\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$$

$$\tan\left(\frac{3}{2}\pi + \theta\right) = -\cot\theta$$

$$\tan\left(\frac{3}{2}\pi - \theta\right) = \cot\theta$$

(2) 덧셈정리

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \quad (1 \mp \tan\alpha\tan\beta \neq 0)$$

(3) 배각공식

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha} \quad (1 - \tan^2\alpha \neq 0)$$

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$$

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$

$$\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha} \quad (1 - 3\tan^2\alpha \neq 0)$$

(4) 반각공식

$$\sin^2 \frac{\alpha}{2} = \frac{1 - \cos\alpha}{2}$$

$$\cos^2 \frac{\alpha}{2} = \frac{1 + \cos\alpha}{2}$$

$$\tan^2 \frac{\alpha}{2} = \frac{1 - \cos\alpha}{1 + \cos\alpha}$$

(5) 곱을 합 또는 차로 고치는 공식

$$\sin\alpha\cos\beta = \frac{1}{2}\{\sin(\alpha+\beta) + \sin(\alpha-\beta)\}$$

$$\cos\alpha\sin\beta = \frac{1}{2}\{\sin(\alpha+\beta) - \sin(\alpha-\beta)\}$$

$$\cos\alpha\cos\beta = \frac{1}{2}\{\cos(\alpha+\beta) + \cos(\alpha-\beta)\}$$

$$\sin\alpha\sin\beta = -\frac{1}{2}\{\cos(\alpha+\beta) - \cos(\alpha-\beta)\}$$

(6) 합 또는 차를 곱으로 고치는 공식

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

(7) 삼각함수의 합성

$$a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}\sin(\theta+\alpha) = \sqrt{a^2+b^2}\cos(\theta-\beta)$$

$$\alpha = \tan^{-1}\left(\frac{b}{a}\right), \quad \beta = \tan^{-1}\left(\frac{a}{b}\right) = \frac{\pi}{2} - \alpha$$

(8) 삼각함수 항등식

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

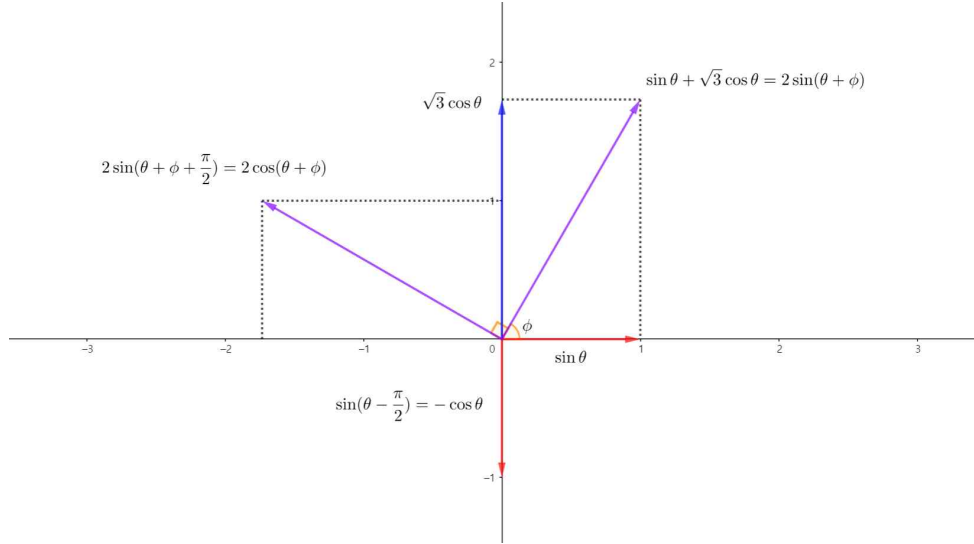
$$\cot^2\theta + 1 = \csc^2\theta$$

(9) 부정적분과 미분

$$\textcircled{1} \frac{d}{dx}\left(\int f(x)dx\right) = f(x)$$

$$\textcircled{2} \int\left(\frac{d}{dx}f(x)\right)dx = f(x) + C$$

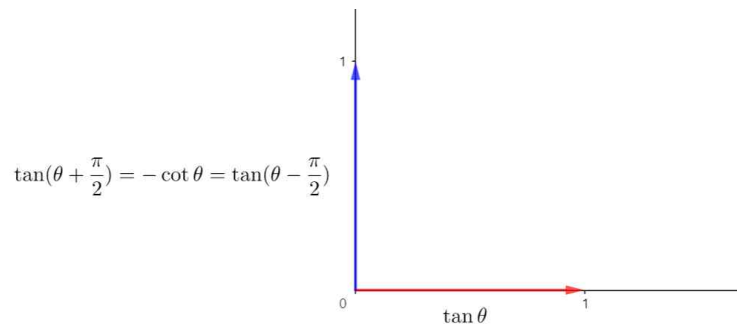
(10) 삼각함수 위상자(Phasor)



- x 축의 양의 방향을 \sin 축, y 축의 양의 방향을 \cos 축으로 설정하여 삼각함수의 위상을 벡터로 표현한다.
- 각이 더해질 경우 위상자는 길이는 유지된 채 반시계방향으로 회전한다.
- 위상자가 표현하는 삼각함수의 계수는 그 위상자의 길이로 표현된다.
- 서로 다른 두 위상자를 벡터합하면 이는 각 위상자가 표현하는 삼각함수의 합성과 같다.

가령, 위 사진에서 $\sin\theta + \sqrt{3}\cos\theta = 2\sin(\theta + \phi)$ 이고 $\phi = \frac{\pi}{3}$ 이다.

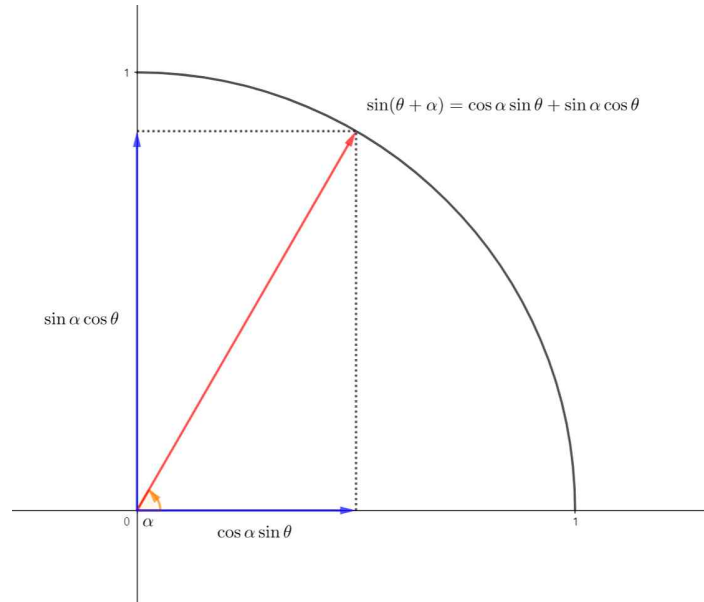
- $\sin\theta$ 위상자를 시계방향으로 $\frac{\pi}{2}$ 만큼 회전시키면 $\sin\left(\theta - \frac{\pi}{2}\right)$ 이며, 이는 $-\cos$ 축이므로 $\sin\left(\theta - \frac{\pi}{2}\right) = -\cos\theta$ 이다.
- \sin 을 \csc 로, \cos 을 \sec 로 바꾸면 \csc 와 \sec 에 대한 각변환이 가능하나 덧셈정리와 합성은 성립하지 않는다. 즉, $\csc\theta + \sqrt{3}\sec\theta = 2\csc(\theta + \phi)$ 는 성립하지 않는다.



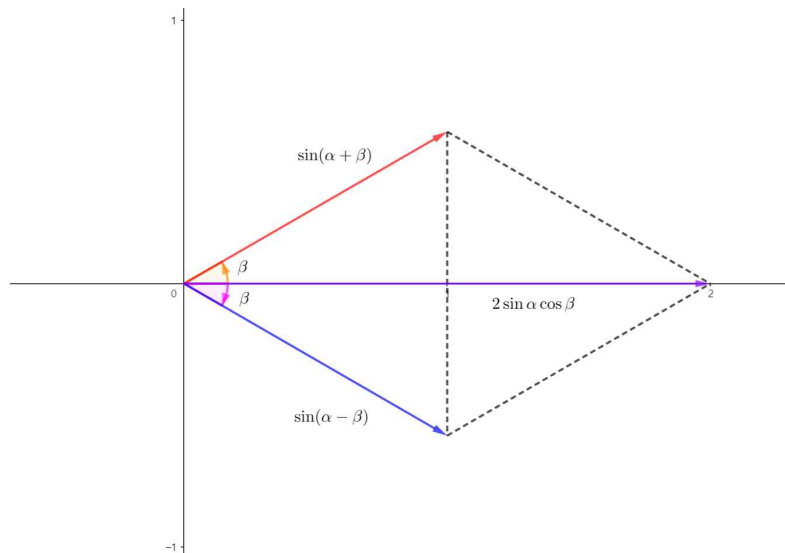
- \tan 와 $-\cot$ 의 경우 위와 같이 xy 평면의 제 1사분면만을 이용하여 도시할 수 있다. 이 경우 주기가 π 이므로 시계방향으로 회전하는 경우 돌아간 각도를 $\frac{3}{2}\pi$ 가 아닌 $\frac{\pi}{2}$ 로 본다.

(11) 삼각함수 위상자의 활용

- 삼각함수 위상자를 사용하면 (1), (2), (5), (6), (7)의 공식들은 모두 증명 가능하다. 이에 대한 예시로 (2)와 (5)의 첫 번째 공식의 증명을 제시해 놓는다.



그림과 같은 단위원에서 $\sin(\theta + \alpha)$ 는 $\sin\theta$ 축에서 길이 1인 위상자가 각도 α 만큼 회전한 것이다. 따라서 위상자의 종점에서 \sin 축, \cos 축에 각각 수선의 발을 내리면 원점을 시점으로 하고 두 수선의 발을 종점으로 하는 두 위상자의 길이는 각각 $\cos\alpha$, $\sin\alpha$ 이다. 즉 처음의 $\sin(\theta + \alpha)$ 위상자(빨간색)를 \sin 축 성분과 \cos 축 성분의 두 위상자(파란색)로 분해할 수 있고 이들의 길이는 각각 $\cos\alpha$, $\sin\alpha$ 이므로, $\sin(\theta + \alpha) = \cos\alpha\sin\theta + \sin\alpha\cos\theta$ 가 성립한다.



그림과 같이 기준각이 α 인 위상 평면에서 $\sin(\alpha + \beta)$, $\sin(\alpha - \beta)$ 는 길이가 1인 $\sin\alpha$ 축 위상자가 각각 β , $-\beta$ 만큼 회전한 것이다. 따라서 이들을 합성한 보라색 위상자는 $\sin\alpha$ 축으로 길이가 $2\cos\beta$ 인 위상자이므로 $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta$ 이고 증명이 완료되었다.

이와 비슷한 방법으로 (1), (2), (5), (6), (7)의 공식들을 모두 증명할 수 있다.

(12) 부정적분의 기본 성질

$$\textcircled{1} \int kf(x)dx = k \int f(x)dx \quad (k \in \mathbb{R})$$

$$\textcircled{2} \int \{f(x) \pm g(x)\}dx = \int f(x)dx \pm \int g(x)dx \quad (\text{복부호동순})$$

(13) 치환적분법

$$\textcircled{1} g(x) = t \text{ 일 때, } \int f(g(x))g'(x)dx = \int f(t)dt$$

$$\textcircled{2} \int f(x)dx = F(x) + C \text{ 일 때, } \int f(ax+b)dx = \frac{1}{a}F(ax+b) + C$$

$$\textcircled{3} \int \frac{1}{ax+b}dx = \frac{1}{a} \ln|ax+b| + C$$

$$\textcircled{4} \int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$$

(14) 바이어슈트라스 치환 (Weierstrass Substitution)

- 바이어슈트라스 치환은 삼각함수의 유리 적분을 유리식의 적분으로 바꿔주는 치환법이다.

$$\tan \frac{x}{2} = t$$

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad \tan x = \frac{2t}{1-t^2}$$

(15) 오일러 치환 (Euler's Substitution)

- 오일러 치환은 유리 이변수 함수 R 에 대하여 다음과 같은 부정적분을 계산하기 위한 치환법이다.

$$\int R(x, \sqrt{ax^2 + bx + c})dx$$

[1] 제 1종 오일러 치환

$a > 0$ 일 때,

$$\sqrt{ax^2 + bx + c} = \pm x\sqrt{a} + t,$$

$$x = \frac{c-t^2}{\pm 2t\sqrt{a-b}}$$

와 같이 치환한다.

[2] 제 2종 오일러 치환

$c > 0$ 일 때,

$$\sqrt{ax^2 + bx + c} = xt \pm \sqrt{c},$$

$$x = \frac{\pm 2t\sqrt{c} - b}{a - t^2}$$

와 같이 치환한다.

[3] 제 3종 오일러 치환

방정식 $ax^2 + bx + c = 0$ 이 두 실근 α, β 를 가질 때,

$$\sqrt{ax^2 + bx + c} = \sqrt{a(x - \alpha)(x - \beta)} = (x - \alpha)t,$$

$$x = \frac{\alpha\beta - \alpha t^2}{a - t^2}$$

와 같이 치환한다.

(16) 부정적분 공식 ($C \in \mathbb{R}$)

1. $\int dx = x + C$

2. $\int adx = ax + C$ ($a \in \mathbb{R}$)

3. $\int x^n dx = \frac{1}{n+1}x^{n+1} + C$ ($-1 \neq n \in \mathbb{R}$)

4. $\int \frac{1}{x} dx = \ln|x| + C$

5. $\int e^x dx = e^x + C$

6. $\int a^x dx = \frac{a^x}{\ln a} + C$ ($0 < a \neq 1$)

7. $\int \ln x dx = x \ln x - x + C$

8. $\int \sin x dx = -\cos x + C$

9. $\int \cos x dx = \sin x + C$

10. $\int \tan x dx = \ln|\sec x| + C$

11. $\int \csc x dx = \ln|\csc x - \cot x| + C$

12. $\int \sec x dx = \ln|\sec x + \tan x| + C$

13. $\int \cot x dx = \ln |\sin x| + C$
14. $\int \sec^2 x dx = \tan x + C$
15. $\int \csc^2 x dx = -\cot x + C$
16. $\int \sec x \tan x dx = \sec x + C$
17. $\int \csc x \cot x dx = -\csc x + C$
18. $\int \frac{1}{(x+a)(x+b)} dx = \frac{1}{b-a} \ln \left| \frac{x+a}{x+b} \right| + C$
19. $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + C$
20. $\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln |x + \sqrt{x^2 \pm a^2}| + C$
21. $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$
22. $\int \sin^{-1} x dx = x \sin^{-1} x + \sqrt{1-x^2} + C$
23. $\int \cos^{-1} x dx = x \cos^{-1} x - \sqrt{1-x^2} + C$
24. $\int \tan^{-1} x dx = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$

(17) 헤비사이드 법 (부분분수 분해)

$$\frac{g(x)}{f(x)} = \frac{g(x)}{(x-a_1)(x-a_2)\cdots(x-a_n)} = \sum_{i=1}^n \frac{b_i}{x-a_i} = \frac{b_1}{x-a_1} + \frac{b_2}{x-a_2} + \cdots + \frac{b_n}{x-a_n} \text{ 일 때}$$

$$\frac{f(x)}{(x-a_i)} = h_i(x) \text{ 라 하면 } b_i = \frac{g(a_i)}{h_i(a_i)} \text{ 가 성립한다. } (1 \leq i \leq n)$$

(18) 부분적분법

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

(19) 삼각함수의 거듭제곱의 부정적분 공식 (Reduction Formula)

$$n \in \mathbb{N} - \{1\},$$

$$\int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx$$

$$\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx$$

$$\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx$$

(20) 이상적분 (Improper Integration)

- 이상적분은 적분구간의 끝 값이 특정 실수값 또는 $\pm\infty$ 로 접근할 때의 정적분이다. (양끝값이 모두 극한으로써 작용할 수도 있다.)

- 즉,

$$\lim_{b \rightarrow \infty} \int_a^b f(x) dx, \quad \lim_{a \rightarrow -\infty} \int_a^b f(x) dx$$

$$\lim_{c \rightarrow b^-} \int_a^c f(x) dx, \quad \lim_{c \rightarrow a^+} \int_c^b f(x) dx$$

등과 같은 정적분이다. (Apostol, T (1967), Calculus, Vol. 1 (2nd ed.), Jon Wiley & Sons.) 또한 기호의 남용(abuse of notation)에 의해 적분구간에 $\pm\infty$ 등을 포함하여 쓰기도 한다.

- 예를 들어, $\int_1^{\infty} \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \left(-\frac{1}{b} + \frac{1}{1}\right) = 1$ 이다.

- 또한, $\frac{1}{\sqrt{x}}$ 의 경우 $x=0$ 에서 정의되지 않지만 구간 $[0, 1]$ 에서의 이상적분은 정의 가능하다. 즉,

$$\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{a \rightarrow 0^+} \int_a^1 \frac{1}{\sqrt{x}} dx = \lim_{a \rightarrow 0^+} (2 - 2\sqrt{a}) = 2$$

이다. 이상적분을 극한으로 변환하여 계산할 때 극한이 발산하면 이상적분이 발산한다고 하고, 대표적인 예시로 $\frac{1}{x}$ 를 0부터 1까지 적분한 이상적분은 발산한다. 이상적분의 정의에 따른 극한이 수렴할 경우 그 수렴값을 이상적분의 값으로 한다.

- 적분구간의 내점에서 함수가 무한대로 발산하는 등 유계가 아닌 구간이 존재하면, 그 점을 기점으로 적분구간을 쪼갠 후 적분을 진행해야 한다. 예를 들어,

$$\begin{aligned} \int_{-1}^1 \frac{1}{\sqrt[3]{x^2}} dx &= \lim_{s \rightarrow 0^-} \int_{-1}^s \frac{1}{\sqrt[3]{x^2}} dx + \lim_{t \rightarrow 0^+} \int_t^1 \frac{1}{\sqrt[3]{x^2}} dx \\ &= \lim_{s \rightarrow 0^-} 3(1 - \sqrt[3]{s}) + \lim_{t \rightarrow 0^+} 3(1 - \sqrt[3]{t}) = 3 + 3 = 6. \end{aligned}$$

- 그러나 이와 비슷한 $\int_{-1}^1 \frac{1}{x} dx$ 는 0을 기점으로 한 각각의 적분이 모두 발산하므로 동일한

논리로 계산할 수 없다. (기함수의 적분을 생각하면 직관적으로 0이겠지만 값이 정의되지 않는다.)

(21) 코시 주요값 (Cauchy Principal Value)

- 오귀스탱 루이 코시가 도입한 코시 주요값은 일반적인 정적분으로 값을 구할 수 없는 일부 이상적분의 값을 구하는 방법 중 하나이다.

[def] 함수 $f : \mathbb{R} \rightarrow \mathbb{R}$ 가 $x = x_0$ 근처에서 발산한다고 하자. 그러면 $a < x_0 < b$ 에서의 적분

$$\int_a^b f(x) dx$$

는 리만 적분 또는 르벡 적분으로서 그 값이 존재하지 않을 수 있다. 그러나 만약 다음과 같은 극한이 수렴한다면, 이를 코시 주요값으로 정의한다.

$$P \int_a^b f(x) dx := \lim_{\epsilon \rightarrow 0^+} \left[\int_a^{x_0 - \epsilon} f(x) dx + \int_{x_0 + \epsilon}^b f(x) dx \right]$$

앞서 언급된 $\int_{-1}^1 \frac{1}{x} dx$ 는 코시 주요값을 적용하면

$$\lim_{a \rightarrow 0^+} \left(\int_{-1}^{-a} \frac{1}{x} dx + \int_a^1 \frac{1}{x} dx \right) = 0$$

과 같이 계산할 수 있다. 코시 주요값은

$$PV \int f(x) dx, \quad \text{p.v.} \int f(x) dx, \quad \int_L^* f(z) dz$$

등으로 표기한다.

(22) Cauchy-Schlömilch Transformation

- Cauchy-Schlömilch Substitution 또는 Cauchy-Schlömilch Transformation은

$$u = x - \frac{1}{x}$$

일 때

$$PV \int_{-\infty}^{\infty} F(u) dx = PV \int_{-\infty}^{\infty} F(x) dx$$

임을 이용하는 치환이다. ($F(u)du$ 가 아니라 $F(u)dx$ 임에 유의하라.)

pf) $u = x - \frac{1}{x}$, $x^2 - ux - 1 = 0$ 에서

$$x_{\pm} = \frac{u \pm \sqrt{u^2 + 4}}{2} \quad (\text{복부호동순})$$

이고

$$\int_{-\infty}^{\infty} F(u)dx = \int_{-\infty}^{0^-} F(u)dx_- + \int_{0^+}^{\infty} F(u)dx_+ = \int_{-\infty}^{\infty} F(u)(x_- + x_+)du$$

이다. 한편

$$x_{\pm}' = \frac{1}{2} \left(1 \pm \frac{u}{\sqrt{u^2 + 4}} \right)$$

이므로 $x_- + x_+ = 1$ 이고

$$\int_{-\infty}^{\infty} F(u)dx = \int_{-\infty}^{\infty} F(u)(x_- + x_+)du = \int_{-\infty}^{\infty} F(u)du$$

이다. (Glasser, M. L. (1983). A remarkable property of definite integrals. mathematics of computation, 561-563.)

(23) Glasser's Master Theorem

- (22)번의 $u = x - \frac{1}{x}$ 를

$$u = x - \sum_{j=1}^{n-1} \frac{a_j}{x - C_j} \quad \dots [1]$$

로 대체해도

$$PV \int_{-\infty}^{\infty} F(u)dx = PV \int_{-\infty}^{\infty} F(x)dx \quad \dots [2]$$

가 성립한다는 것이 알려져 있다. (여기서 $\{a_j\}$ 는 양의 실수열, C_j 는 실수이다.)

pf) 일반성을 잃지 않고 $C_1 < C_2 < C_3 \dots$ 라 하자. 이때 식 [1]은 x 에 대하여 다음과 같은 n 차식으로 환원된다.

$$x^n - \left(u - \sum_{j=1}^{n-1} C_j\right)x^{n-1} + \dots = 0$$

대수학의 기본정리에 의해 이 방정식은 복소 범위에서 (중복을 포함하여) n 개의 근을 가지고, n 개의 근 $x_i \in \mathbb{C}$ ($1 \leq i \leq n$)에 대하여 근과 계수의 관계에 의해

$$x_1 + x_2 + \dots + x_n = u - \sum_{j=1}^{n-1} C_j$$

가 성립한다. 따라서

$$x_1' + x_2' + \dots + x_n' = 1$$

이고,

$$\begin{aligned} I &= \left(\int_{-\infty}^{C_1^-} dx_1 + \int_{C_1^+}^{C_2^-} dx_2 + \dots + \int_{C_{n-1}^+}^{\infty} dx_n \right) F(u) \\ &= \int_{-\infty}^{\infty} F(u)(x_1' + x_2' + \dots + x_n') du = \int_{-\infty}^{\infty} F(u) du \end{aligned}$$

이다. 이와 같은 논리를 그대로 적용하면 다음과 같은 치환을 적용해도 식 [2]가 성립한다.

$$u = x - \sum_j a_j \cot[(x - C_j)^{-1}]$$

위 정리를 이용하면 특정한 값으로 수렴하는 아주 복잡한 적분식을 만들어낼 수 있다.

$$\int_{-\infty}^{\infty} \frac{1}{u^2 + 1} du = [\tan^{-1} u]_{-\infty}^{\infty} = \pi$$

이므로

$$u = x - \frac{1}{x} - \frac{1}{x+1} - \frac{1}{x+2}$$

를 대입하여 Glasser's Master Theorem을 적용하면

$$\int_{-\infty}^{\infty} \frac{1}{\left(x - \frac{1}{x} - \frac{1}{x+1} - \frac{1}{x+2}\right)^2 + 1} dx = \pi$$

이고, 식을 전개하여 짝수차항만 추출하면 다음의 정적분을 얻는다.

$$\int_0^{\infty} \frac{x^{14} - 15x^{12} + 82x^{10} - 190x^8 + 184x^6 - 60x^4 + 16x^2}{x^{16} - 20x^{14} + 156x^{12} - 616x^{10} + 1388x^8 - 1792x^6 + 1152x^4 - 224x^2 + 16} dx = \frac{\pi}{2}$$

이와 같이 복잡한 정적분이 주어졌을 때 Glasser's Master Theorem을 이용하기 위해 식을 변형하여 역추적을 하는 것도 하나의 방법이다. (이를 이용하지 않는다면 복소적분과 매우 복잡한 계산과정을 거쳐야할 것이다.)

예제) Glasser's Master Theorem을 이용하여 다음 식이 성립함을 증명하시오.

$$\int_0^{\infty} \frac{x^8 - 4x^6 + 9x^4 - 5x^2 + 1}{x^{12} - 10x^{10} + 37x^8 - 42x^6 + 26x^4 - 8x^2 + 1} dx = \frac{\pi}{2}$$

Glasser, M. L. (1983). A remarkable property of definite integrals. mathematics of computation, 561-563.

(24) 정적분 테크닉

① 적분 구간을 이용한 치환 ($x \mapsto a+b-x$)

$$\int_a^b f(x) dx = \int_b^a f(a+b-x)(-dx) = \int_a^b f(a+b-x) dx$$

- $f(x) + f(a+b-x)$ 의 정적분이 쉽게 계산되는 경우

$$\int_a^b f(x) dx = \int_a^b \frac{f(x) + f(a+b-x)}{2} dx$$

를 이용하여 정적분을 계산할 수 있다.

② 대칭 치환 ($x \mapsto -x$)

$$\int_{-c}^c f(x) dx = \int_c^{-c} f(-x)(-dx) = \int_{-c}^c f(-x) dx$$

- 이는 ①에서 $a+b=0$ 인 특수한 경우이지만, 마찬가지로 자주 등장한다.

$$\int_{-c}^c f(x)dx = \int_{-c}^c \frac{f(x)+f(-x)}{2} dx$$

(25) 파인만 적분 테크닉 (The Feynman Integration Technique)

- 다음과 같은 적분을 생각하자.

$$I = \int e^{ax} dx = \frac{1}{a} e^{ax} + C$$

여기서 x 는 적분변수이고, a 는 상수이다. 따라서 x 에 대해 미분 또는 적분을 할 때 a 는 관여하지 않는다. 하지만 관점을 조금 바꿔 보면 피적분함수인 e^{ax} 를

$$f(x, a) = e^{ax}$$

인 이변수함수로 볼 수도 있을 것이다. 이 관점에 따라 위 적분의 양변을 a 에 대하여 미분 (즉, 편미분) 해보면,

$$\frac{d}{da} \int e^{ax} dx = \frac{\partial}{\partial a} \left[\frac{1}{a} e^{ax} + C \right] = -\frac{1}{a^2} e^{ax} + \frac{x}{a} e^{ax} = \frac{(ax-1)e^{ax}}{a^2} \dots [3]$$

이다. 한편

$$\frac{d}{da} \int e^{ax} dx = \int \left[\frac{\partial}{\partial a} e^{ax} \right] dx = \int x e^{ax} dx$$

이고 실제로 이는 [3]의 결과와 일치한다. 리처드 파인만에 의해 유명해진 이 적분법은 다음과 같은 라이프니츠 적분 규칙(Leibniz Integral Rule)의 특수한 경우이다.

$$\frac{\partial}{\partial z} \int_{a(z)}^{b(z)} f(x, z) dx = \int_{a(z)}^{b(z)} \frac{\partial f}{\partial z} dx + f(b(z), z) \frac{\partial b}{\partial z} - f(a(z), z) \frac{\partial a}{\partial z}$$

즉, 함수 $f(x, \alpha)$ 가 α 에 대해 미분가능하고 $\frac{\partial f}{\partial \alpha}$ 가 연속함수이면 다음이 성립한다.

$$\frac{d}{d\alpha} \int_a^b f(x, \alpha) dx = \int_a^b \frac{\partial}{\partial \alpha} f(x, \alpha) dx$$

이 테크닉을 이용하면 다음과 같은 특이한 정적분의 값을 구할 수 있다.

$$\phi(\alpha) = \int_0^\pi \ln(1 - 2\alpha \cos x + \alpha^2) dx = 2\pi \ln |\alpha| \quad (|\alpha| > 1)$$

이 테크닉을 사용하기 위해서는 피적분함수를 강제적으로 이변수함수로 바꿔야 하므로, 특정 숫자를 α 로 바꾸어 매개변수를 강제로 추가하거나 $e^{-\alpha x}$ 등을 곱해 이변수함수로 만들어준 후 $\alpha=0$ 일 때의 값을 구하는 식으로 처리한다.

또한 피적분식의 결과를 α 에 대하여 미분한 값이 α 에 대한 함수로 나오므로 이는 α 에 대한 미분방정식을 푸는 것으로 이어진다. (복소적분 외에) 파인만 테크닉을 이용해야 하는 문제들은 그 미분방정식의 해법이 어렵지 않은 것들만 수록하였다.

[1] Feynman, R. P. "A Different Set of Tools." In 'Surely You're Joking, Mr. Feynman!': Adventures of a Curious Character. New York: W. W. Norton, 1997.

[2] Hijab, O. Introduction to Calculus and Classical Analysis. New York: Springer-Verlag, p. 189, 1997.

[3] Woods, F. S. "Differentiation of a Definite Integral." §60 in Advanced Calculus: A Course Arranged with Special Reference to the Needs of Students of Applied Mathematics. Boston, MA: Ginn, pp. 141-144, 1926.

(26) 역함수의 적분 공식

- 실함수 f 와 그 역함수 f^{-1} , $F(x) = \int f(x)dx$ 에 대하여 다음이 성립한다. (양변을 미분한 후 chain rule을 적용하면 쉽게 증명된다.)

$$\int f^{-1}(x)dx = x \cdot f^{-1}(x) - F(f^{-1}(x)) + C$$

※ 적분구간에 $\pm\infty$ 또는 그 지점에서 함숫값이 정의되지 않는 값이 포함된 경우, 즉 이상적분의 경우 비록 교과 외 과정이긴 하나 (20)의 설명만으로도 충분히 풀 수 있을 것이라 생각하여 따로 표시하지 않았습니다. 간혹 $\exp x$ 라는 표기가 등장하는데, 이는 e^x 와 같으며 x 자리에 복잡한 분수식이 들어갔을 때 식이 너무 작아져 가독성이 떨어지는 것을 방지하기 위한 표기법입니다.

※※ 쌍곡선함수($\sinh x$, $\cosh x$, $\tanh x$)와 그 역함수는 등장하지 않으며, 역쌍곡선함수의 경우 $\tanh^{-1}x$ 라는 표기 대신 $\frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|$ 등으로 나타내었습니다. 적분 결과 또는 피적분수에 역삼각함수가 등장하는 경우 고등학교 교육과정을 벗어나는 것이기는 하나, (16)의 19, 21번 공식을 사용하는 수준에서 모두 해결 가능합니다.

※※※ 위에서 언급된 테크닉과 고등학교 미적분 수준을 벗어나는 문제는 없습니다. 즉 모든 문제를 고등학교 미적분 교과 내용과 위에서 언급된 테크닉을 이용하여 해결할 수 있고, 유수 정리(Residue Theorem)와 경로 적분(Contour Integration) 등 복소해석학의 정리가 사용될 일은 없습니다.

※※※※ 해설을 보아도 알 수 있겠지만, 모든 문제는 정확한 수치를 구할 수 있고, 부정적분과 달리 그 값이 단 하나로 특정됩니다.

[1~200] 다음 정적분의 값을 구하시오.

1. $\int_0^{\pi} \frac{x \sin x}{3 + \sin^2 x} dx$

2. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx$

3. $\int_0^{\frac{\pi}{2}} \frac{\cos^{\sqrt{5}} x}{\sin^{\sqrt{5}} x + \cos^{\sqrt{5}} x} dx$

4. $\int_0^{\frac{\pi}{4}} \frac{\sin x}{\sin x + \cos x} dx$

5. $\int_0^8 \frac{x^3 - 2x + 1}{\sqrt[3]{x}} dx$

6. $\int_{-3}^3 \frac{2x^2}{2^x + 1} dx$

7. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{e^{1/x} + 1} dx$

8. $\int_0^{\frac{\pi}{2}} \sin 3x \cos 4x dx$

9. $\int_0^{\frac{\pi}{2}} \sqrt[3]{\tan x} dx$

10. $\int_0^{\frac{\pi}{4}} \sqrt[3]{\tan x} dx$

11. $\int_0^1 \frac{1}{x^x} dx$ (급수의 형태로 나타내어도 무방함.)

12. $\int_0^1 \frac{1}{1 + \left(1 - \frac{1}{x}\right)^{2022}} dx$

13. $\int_{-\infty}^{\infty} e^{-x^2} dx$

14. $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$

15. $\int_0^{\infty} 4\pi r^2 |\psi(r)|^2 dr$ ($\psi(r) = \frac{1}{\sqrt{\pi a_0^3}} e^{-\frac{r}{a_0}}$)
16. $\int_{\frac{a_0}{2}}^{\infty} 4\pi r^2 |\psi(r)|^2 dr$ ($\psi(r) = \frac{1}{\sqrt{\pi a_0^3}} e^{-\frac{r}{a_0}}$)
17. $\int_0^{\frac{\pi}{2}} \sin^{10} x dx$
18. $\int_0^1 \frac{x^{2022} - 1}{\ln x} dx$
19. $\int_0^1 \frac{\ln x}{x-1} dx$
20. $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx$
21. $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx$
22. $\int_0^1 \sqrt[3]{2x^3 - 3x^2 - x + 1} dx$
23. $\int_0^1 \frac{\ln(x+1)}{x^2+1} dx$
24. $\int_0^2 \frac{\ln(x+1)}{x^2-x+1} dx$
25. $\int_0^1 \frac{\ln(x^2+1)}{x+1} dx$
26. $\int_0^{\frac{\pi}{2}} \ln(\cos x) dx$
27. $\int_0^1 \frac{\ln x}{x+1} dx$
28. $\int_0^{\frac{\pi}{2}} \cos^{2022} x \cos 2022x dx$
29. $\int_0^1 (\ln x)^{2022} dx$
30. $\int_1^e \frac{\ln x}{\sqrt{x}} dx$
31. $\int_0^{\frac{\pi}{2}} \ln(2 + \tan^2 x) dx$
32. $\int_0^{\infty} \frac{x - \sin x}{x^3(x^2+4)} dx$
33. $\int_0^{\pi} \frac{1 - \sin x}{1 + \sin x} dx$

34. $\int_0^{\frac{3}{5}} \frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}} dx$
35. $\int_0^{\frac{\pi}{2}} \sin 2022x \cdot \sin^{2020} x dx$
36. $\int_1^{\sqrt{2}} \frac{x^4-1}{x^2\sqrt{x^4-x^2+1}} dx$
37. $\int_0^{\frac{\pi}{3}} \frac{\sin 2x}{2+\cos x} dx$
38. $\int_0^{\frac{\pi}{4}} \tan^9 x dx$
39. $\int_0^1 \frac{x-1}{(1+x^3)\ln x} dx$
40. $\int_0^{\frac{\pi}{2}} \frac{(1+\sec^2 t)\sqrt{\sec t}}{(1+\sec t)^2-2} dt$
41. $\int_0^{\frac{\pi}{2}} \frac{\cos x}{2-\sin 2x} dx$
42. $\int_0^{\infty} (\ln(e^x+1)-x) dx$
43. $\int_0^{\infty} \frac{x}{e^x+1} dx$
44. $\int_0^{\frac{\pi}{2}} \frac{\ln(\tan x)}{1-\tan x+\tan^2 x} dx$
45. $\int_{-\infty}^{\infty} \frac{\cos\left(x-\frac{1}{x}\right)}{1+x^2} dx$
46. $\int_0^{\infty} \frac{\cos x}{x^2+1} dx$
47. $\int_0^{\infty} \frac{x \sin x}{x^2+1} dx$
48. $\int_0^{\infty} \frac{\ln(1+x)}{x(x^2+1)} dx$
49. $\int_0^{\frac{\pi}{2}} \frac{\sin(u+\sqrt{\pi} \tan u)}{\sin u} du$
50. $\int_0^{\pi} \sin^2(x-\sqrt{\pi^2-x^2}) dx$
51. $\int_0^{\infty} e^{-x^2} \cos 2x dx$
52. $\int_0^{\frac{\pi}{2}} \cos(\tan x) dx$

53. $\int_0^{\frac{\pi}{2}} \frac{x}{\tan x} dx$
54. $\int_0^{\pi} \ln(1 - 2e \cos x + e^2) dx$
55. $\int_0^{\pi} \ln(1 + \sin^2 t) dt$
56. $\int_0^4 \frac{\ln x}{\sqrt{4x - x^2}} dx$
57. $\int_0^{\infty} \frac{1}{(1 + x^2)(1 + x^{\pi})} dx$
58. $\int_0^{\infty} \frac{1}{(1 + x)(\pi^2 + (\ln x)^2)} dx$
59. $\int_0^{\infty} \frac{x - 1}{\sqrt{2^x - 1} \ln(2^x - 1)} dx$
60. $\int_0^{3\pi} \frac{1}{\sin^4 x + \cos^4 x} dx$
61. $\int_0^{\pi} \cos^4 x dx$
62. $\int_0^{\frac{\pi}{2}} \sin^6 x \cos^3 x dx$
63. $\int_0^{\frac{\pi}{4}} \sin^3 x \sec^2 x dx$
64. $\int_1^3 \frac{1}{x^2 \sqrt{x^2 + 4}} dx$
65. $\int_{\sqrt{3}-1}^{2\sqrt{2}-1} \frac{1}{(x+1)\sqrt{x^2+2x+2}} dx$
66. $\int_0^{\tan^{-1}(\sqrt{6})} \frac{1}{1 + \cos^2 x} dx$
67. $\int_0^{\frac{\pi}{2}} \frac{1}{1 + \cos x} dx$
68. $\int_{\ln(e-1)}^{\ln(e^3-1)} \frac{e^x \ln(e^x + 1)}{e^x + 1} dx$
69. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{\sin x} dx$
70. $\int_0^{\frac{\pi}{4}} \frac{1}{1 - 3\cos^2 x} dx$
71. $\int_{-1}^0 \frac{x^3 - x - 2}{x^3 - x^2 + x - 1} dx$

72. $\int_3^4 \frac{x+4}{x^3+3x^2-10x} dx$
73. $\int_5^6 \frac{7x^3-13x^2-24x+24}{x^4-3x^3-10x^2+24x} dx$
74. $\int_1^2 \frac{1}{x^2\sqrt{2x-x^2}} dx$
75. $\int_1^e \frac{x^4+81}{x(x^2+9)^2} dx$
76. $\int_0^3 \sqrt{\frac{4-x}{x}} dx$
77. $\int_0^{\frac{1}{\sqrt[3]{2}}} \sqrt{\frac{x}{1-x^3}} dx$
78. $\int_0^{\frac{3}{4}} \sqrt{x} \sqrt{1-x} dx$
79. $\int_0^{\tan^{-1}(\sqrt[3]{3})} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$
80. $\int_0^{\frac{\pi}{4}} \tan^4 x \sec^4 x dx$
81. $\int_2^4 \frac{x\sqrt{x-1}}{x-\sqrt{x}} dx$
82. $\int_0^{\ln 3} \frac{1}{\sqrt{1+e^x}} dx$
83. $\int_1^2 \frac{x^2+2x-1}{2x^3+3x^2-2x} dx$
84. $\int_0^{\frac{\pi}{3}} 13e^{2x} \cos 3x dx$
85. $\int_0^{\pi} e^{-x} \sin^2 2x dx$
86. $\int_0^{\frac{\pi}{4}} \sqrt{\tan x} dx$
87. $\int_0^{\frac{\pi}{4}} \sqrt{\tan x} \sqrt{1-\tan x} dx$
88. $\int_1^{\infty} \frac{(\ln x)^{627}}{x^{2022}} dx$
89. $\int_0^{\sqrt{6}} \cos^2(\tan^{-1}(\sin(\cot^{-1} x))) dx$
90. $\int_0^{\frac{\pi}{6}} \frac{\sec^2 x}{(\sec x + \tan x)^{5/2}} dx$

91. $\int_1^2 \frac{1}{x\sqrt{x^2+4x-4}} dx$
92. $\int_1^{\frac{25}{73}} \frac{1}{x\sqrt{-x^2+x+2}} dx$
93. $\int_1^2 \frac{x^2}{\sqrt{-x^2+3x-2}} dx$
94. $\int_0^{\frac{\pi}{4}} \frac{\sin^3(\theta/2)}{\cos(\theta/2) \cdot \sqrt{\cos^3\theta + \cos^2\theta + \cos\theta}} d\theta$
95. $\int_0^{\frac{\pi}{6}} \frac{\tan^4\theta}{1-\tan^2\theta} d\theta$
96. $\int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx$
97. $\int_0^{\infty} \exp\left(-x^2 - \frac{1}{x^2}\right) dx$
98. $\int_0^1 \frac{1-x^{99}}{(1+x)(1+x^{100})} dx$
99. $\int_{-\infty}^{\infty} \exp\left(-\frac{(x^2-13x-1)^2}{611x^2}\right) dx$
100. $\int_0^{\frac{\pi}{2}} \sqrt[n]{\tan x} dx \quad (2 \leq n \in \mathbb{N})$

[해설]

$$1. I = \int_0^{\pi} \frac{x \sin x}{3 + \sin^2 x} dx = \int_0^{\pi} \frac{(\pi - x) \sin x}{3 + \sin^2 x} dx \quad (x \mapsto \pi - x)$$

$$2I = \int_0^{\pi} \left(\frac{x \sin x}{3 + \sin^2 x} + \frac{(\pi - x) \sin x}{3 + \sin^2 x} \right) dx = \int_0^{\pi} \frac{\pi \sin x}{3 + \sin^2 x} dx = \int_0^{\pi} \frac{\pi \sin x}{4 - \cos^2 x} dx$$

$$= \int_1^{-1} \frac{\pi}{u^2 - 4} du \quad (u = \cos x, \quad du = -\sin x dx)$$

$$= \pi \left[\frac{1}{4} \ln \left| \frac{u-2}{u+2} \right| \right]_1^{-1} = \frac{\pi}{2} \ln 3$$

$$\therefore I = \frac{\pi}{4} \ln 3 \quad \blacksquare$$

$$2. I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx$$

$$\frac{\pi}{2} - 2I = \int_0^{\frac{\pi}{2}} dx - \int_0^{\frac{\pi}{2}} \frac{2 \sin x}{\sin x + \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{\sin x + \cos x} dx = [\ln |\sin x + \cos x|]_0^{\frac{\pi}{2}} = 0$$

$$\therefore I = \frac{\pi}{4} \quad \blacksquare$$

$$3. I = \int_0^{\frac{\pi}{2}} \frac{\cos^{\sqrt{5}} x}{\sin^{\sqrt{5}} x + \cos^{\sqrt{5}} x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin^{\sqrt{5}} x}{\sin^{\sqrt{5}} x + \cos^{\sqrt{5}} x} dx \quad (x \mapsto \frac{\pi}{2} - x)$$

$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{\cos^{\sqrt{5}} x}{\sin^{\sqrt{5}} x + \cos^{\sqrt{5}} x} + \frac{\sin^{\sqrt{5}} x}{\sin^{\sqrt{5}} x + \cos^{\sqrt{5}} x} \right) dx = \frac{\pi}{2}$$

$$\therefore I = \frac{\pi}{4} \quad \blacksquare$$

$$4. \int_0^{\frac{\pi}{4}} \frac{\sin x}{\sin x + \cos x} dx = \int_0^{\frac{\pi}{4}} \frac{\sin\left(\frac{\pi}{4} - x\right)}{\sin\left(\frac{\pi}{4} - x\right) + \cos\left(\frac{\pi}{4} - x\right)} dx \quad (x \mapsto \frac{\pi}{4} - x)$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{\cos x - \sin x}{\cos x} dx = \frac{1}{2} \int_0^{\frac{\pi}{4}} (1 - \tan x) dx = \frac{1}{2} [x + \ln |\cos x|]_0^{\frac{\pi}{4}} = \frac{\pi}{8} - \frac{1}{4} \ln 2 \quad \blacksquare$$

5. $t = \sqrt[3]{x}$, $dx = 3t^2 dt$

$$\int_0^8 \frac{x^3 - 2x + 1}{\sqrt[3]{x}} dx = \int_0^2 3t(t^9 - 2t^3 + 1) dt = \left[\frac{3}{11} t^{11} - \frac{6}{5} t^5 + \frac{3}{2} t^2 \right]_0^2$$

$$= \left[\frac{3}{11} x^{\frac{11}{3}} - \frac{6}{5} x^{\frac{5}{3}} + \frac{3}{2} x^{\frac{2}{3}} \right]_0^8 = \frac{28938}{55} \quad \blacksquare$$

6. $I = \int_{-3}^3 \frac{2x^2}{2^x + 1} dx = \int_{-3}^3 \frac{2x^2}{2^{-x} + 1} dx = \int_{-3}^3 \frac{2^x \cdot 2x^2}{2^x + 1} dx$ ($x \mapsto -x$)

$$2I = \int_{-3}^3 \left(\frac{2x^2}{2^x + 1} + \frac{2^x \cdot 2x^2}{2^x + 1} \right) dx = \int_{-3}^3 2x^2 dx = \left[\frac{2}{3} x^3 \right]_{-3}^3 = 36$$

$$\therefore I = \int_{-3}^3 \frac{2x^2}{2^x + 1} dx = 18 \quad \blacksquare$$

7. $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{e^{1/x} + 1} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{e^{-1/x} + 1} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{1/x} \cdot \cos x}{e^{1/x} + 1} dx$ ($x \mapsto -x$)

$$2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\cos x}{e^{1/x} + 1} + \frac{e^{1/x} \cdot \cos x}{e^{1/x} + 1} \right) dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x dx = 2$$

$$\therefore I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{e^{1/x} + 1} dx = 1 \quad \blacksquare$$

8. $\int_0^{\frac{\pi}{2}} \sin 3x \cos 4x dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} (\sin 7x - \sin x) dx = \left[\frac{1}{2} \cos x - \frac{1}{14} \cos 7x \right]_0^{\frac{\pi}{2}} = -\frac{3}{7} \quad \blacksquare$

9. [1] $t^3 = \tan x$, $x = \tan^{-1}(t^3)$, $dx = \frac{3t^2}{1+t^6} dt$

[2] $a = t^2$, $da = 2t dt$, $a = \tan^{2/3} x$

$$\int_0^{\frac{\pi}{2}} \sqrt[3]{\tan x} dx = \int_0^{\infty} \frac{3t^3}{1+t^6} dt \quad \dots \quad [1]$$

$$\begin{aligned}
&= \frac{3}{2} \int_0^\infty \frac{a}{1+a^3} da \quad \dots [2] \\
&= \frac{3}{2} \int_0^\infty \left(-\frac{1}{3(a+1)} + \frac{a+1}{3(a^2-a+1)} \right) da \\
&= \frac{1}{2} \int_0^\infty \left(\frac{2a-1}{2(a^2-a+1)} + \frac{3}{2(a^2-a+1)} - \frac{1}{a+1} \right) da \\
&= \left[\frac{1}{4} \ln \frac{a^2-a+1}{(a+1)^2} + \frac{\sqrt{3}}{2} \tan^{-1} \left(\frac{2}{\sqrt{3}} \left(a - \frac{1}{2} \right) \right) \right]_0^\infty \\
&= \left[\frac{1}{4} \ln \frac{\tan^{4/3} x - \tan^{2/3} x + 1}{(\tan^{2/3} x + 1)^2} + \frac{\sqrt{3}}{2} \tan^{-1} \left(\frac{2}{\sqrt{3}} \left(\tan^{2/3} x - \frac{1}{2} \right) \right) \right]_0^{\frac{\pi}{2}} \\
&= \lim_{x \rightarrow \frac{\pi}{2}} \left\{ \frac{1}{4} \ln \frac{\tan^{4/3} x - \tan^{2/3} x + 1}{(\tan^{2/3} x + 1)^2} + \frac{\sqrt{3}}{2} \tan^{-1} \left(\frac{2}{\sqrt{3}} \left(\tan^{2/3} x - \frac{1}{2} \right) \right) \right\} - \frac{\sqrt{3}}{2} \tan^{-1} \left(-\frac{1}{\sqrt{3}} \right) \\
&= \lim_{t \rightarrow \infty} \left\{ \frac{1}{4} \ln \frac{t^{4/3} - t^{2/3} + 1}{(t^{2/3} + 1)^2} + \frac{\sqrt{3}}{2} \tan^{-1} \left(\frac{2}{\sqrt{3}} \left(t^{2/3} - \frac{1}{2} \right) \right) \right\} + \frac{\sqrt{3}}{12} \pi \\
&= \frac{1}{4} \ln 1 + \frac{\sqrt{3}}{4} \pi + \frac{\sqrt{3}}{12} \pi = \frac{\pi}{\sqrt{3}} \quad \blacksquare
\end{aligned}$$

10. [1] $t^3 = \tan x$, $x = \tan^{-1}(t^3)$, $dx = \frac{3t^2}{1+t^6} dt$

[2] $a = t^2$, $da = 2t dt$, $a = \tan^{2/3} x$

$$\int_0^{\frac{\pi}{4}} \sqrt[3]{\tan x} dx = \int_0^1 \frac{3t^3}{1+t^6} dt \quad \dots [1]$$

$$= \frac{3}{2} \int_0^1 \frac{a}{1+a^3} da \quad \dots [2]$$

$$= \frac{3}{2} \int_0^1 \left(-\frac{1}{3(a+1)} + \frac{a+1}{3(a^2-a+1)} \right) da$$

$$\begin{aligned}
&= \frac{1}{2} \int_0^1 \left(\frac{2a-1}{2(a^2-a+1)} + \frac{3}{2(a^2-a+1)} - \frac{1}{a+1} \right) da \\
&= \left[\frac{1}{4} \ln \frac{a^2-a+1}{(a+1)^2} + \frac{\sqrt{3}}{2} \tan^{-1} \left(\frac{2}{\sqrt{3}} \left(a - \frac{1}{2} \right) \right) \right]_0^1 \\
&= \left[\frac{1}{4} \ln \frac{\tan^{4/3} x - \tan^{2/3} x + 1}{(\tan^{2/3} x + 1)^2} + \frac{\sqrt{3}}{2} \tan^{-1} \left(\frac{2}{\sqrt{3}} \left(\tan^{2/3} x - \frac{1}{2} \right) \right) \right]_0^{\frac{\pi}{4}} \\
&= \left(-\frac{1}{2} \ln 2 + \frac{\sqrt{3}}{12} \pi \right) - \left(0 - \frac{\sqrt{3}}{12} \pi \right) = \frac{\sqrt{3}}{6} \pi - \frac{1}{2} \ln 2 \blacksquare
\end{aligned}$$

$$\begin{aligned}
11. \quad \int_0^1 \frac{1}{x^x} dx &= \int_0^1 e^{-x \ln x} dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-x \ln x)^n}{n!} dx = \sum_{n=0}^{\infty} \int_0^1 \frac{(-x \ln x)^n}{n!} dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^n (\ln x)^n dx
\end{aligned}$$

한편

$$\begin{aligned}
\int_0^1 x^n (\ln x)^n dx &= \left[\frac{1}{n+1} x^{n+1} (\ln x)^n \right]_0^1 - \int_0^1 \frac{1}{n+1} x^{n+1} \cdot \frac{n(\ln x)^{n-1}}{x} dx \\
&= -\frac{n}{n+1} \int_0^1 x^n (\ln x)^{n-1} dx = (-1)^2 \frac{n(n-1)}{(n+1)^2} \int_0^1 x^n (\ln x)^{n-2} dx \\
&= (-1)^3 \frac{n(n-1)(n-2)}{(n+1)^3} \int_0^1 x^n (\ln x)^{n-3} dx = \dots \\
&= (-1)^n \frac{n!}{(n+1)^n} \int_0^1 x^n (\ln x)^{n-n} dx = (-1)^n \frac{n!}{(n+1)^{n+1}}
\end{aligned}$$

이므로

$$\therefore \int_0^1 \frac{1}{x^x} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^n (\ln x)^n dx = \sum_{n=0}^{\infty} (n+1)^{-(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n^n} \blacksquare$$

$$12. \quad I = \int_0^1 \frac{1}{1 + \left(1 - \frac{1}{x}\right)^{2022}} dx = \int_0^1 \frac{1}{1 + \left(\frac{x-1}{x}\right)^{2022}} dx = \int_0^1 \frac{x^{2022}}{x^{2022} + (x-1)^{2022}} dx$$

$$\begin{aligned}
&= \int_0^1 \frac{(1-x)^{2022}}{x^{2022} + (x-1)^{2022}} dx \quad (x \mapsto 1-x) \\
&= \frac{1}{2} \int_0^1 \left(\frac{x^{2022}}{x^{2022} + (1-x)^{2022}} + \frac{(1-x)^{2022}}{x^{2022} + (1-x)^{2022}} \right) dx = \frac{1}{2} \blacksquare
\end{aligned}$$

13. $I = \int_{-\infty}^{\infty} e^{-x^2} dx = 2 \int_0^{\infty} e^{-x^2} dx$ (Gaussian Integral, 가우스 적분)

$$\begin{aligned}
I^2 &= 4 \int_0^{\infty} e^{-x^2} dx \cdot \int_0^{\infty} e^{-y^2} dy = 4 \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy \\
&= 4 \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-r^2} r dr d\theta \quad (r = x^2 + y^2, \theta = \tan^{-1}\left(\frac{y}{x}\right), dx dy = r dr d\theta) \\
&= 4 \int_0^{\frac{\pi}{2}} d\theta \cdot \int_0^{\infty} r e^{-r^2} dr = 2\pi \left[-\frac{1}{2} e^{-r^2} \right]_0^{\infty} = \pi
\end{aligned}$$

$$\therefore I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \blacksquare$$

14. $z = \frac{x-\mu}{\sigma}$, $dz = \frac{dx}{\sigma}$ ($N(\mu, \sigma^2)$ 을 따르는 정규분포의 확률밀도함수)

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} dy \quad (z = \sqrt{2}y, dz = \sqrt{2}dy) \\
&= 1 \blacksquare
\end{aligned}$$

15. $\int_0^{\infty} 4\pi r^2 |\psi(r)|^2 dr = \frac{4}{a_0^3} \int_0^{\infty} r^2 e^{-\frac{2r}{a_0}} dr = \frac{4}{a_0^3} \left\{ \left[-\frac{a_0}{2} r^2 e^{-\frac{2r}{a_0}} \right]_0^{\infty} + \frac{a_0}{2} \int_0^{\infty} 2re^{-\frac{2r}{a_0}} dr \right\}$

$$= \frac{4}{a_0^3} \left\{ 0 + a_0 \left[-\frac{a_0}{2} r e^{-\frac{2r}{a_0}} \right]_0^{\infty} + \frac{a_0^2}{2} \int_0^{\infty} e^{-\frac{2r}{a_0}} dr \right\}$$

$$= \frac{4}{a_0^3} \left\{ \frac{a_0^2}{2} \left[-\frac{a_0}{2} e^{-\frac{2r}{a_0}} \right]_0^\infty \right\} = \frac{4}{a_0^3} \left(\frac{a_0^3}{4} \right) = 1 \blacksquare$$

$\psi(r)$ 은 거리 r 에 따른 수소원자의 바닥상태 파동함수이다. 위 적분은 수소원자의 전구간에서 전자가 발견될 확률이 1임을 의미한다.

$$\begin{aligned} 16. \int_{\frac{a_0}{2}}^\infty 4\pi r^2 |\psi(r)|^2 dr &= \frac{4}{a_0^3} \int_{\frac{a_0}{2}}^\infty r^2 e^{-\frac{2r}{a_0}} dr = \frac{4}{a_0^3} \left\{ \left[-\frac{a_0}{2} r^2 e^{-\frac{2r}{a_0}} \right]_{\frac{a_0}{2}}^\infty + \frac{a_0}{2} \int_{\frac{a_0}{2}}^\infty 2re^{-\frac{2r}{a_0}} dr \right\} \\ &= \frac{4}{a_0^3} \left\{ \frac{a_0^3}{8e} + a_0 \left[-\frac{a_0}{2} r e^{-\frac{2r}{a_0}} \right]_{\frac{a_0}{2}}^\infty + \frac{a_0^2}{2} \int_{\frac{a_0}{2}}^\infty e^{-\frac{2r}{a_0}} dr \right\} \\ &= \frac{4}{a_0^3} \left\{ \frac{3a_0^3}{8e} + \frac{a_0^2}{2} \left[-\frac{a_0}{2} e^{-\frac{2r}{a_0}} \right]_{\frac{a_0}{2}}^\infty \right\} = \frac{4}{a_0^3} \left(\frac{5a_0^3}{8e} \right) = \frac{5}{2e} \simeq 91.97\% \blacksquare \end{aligned}$$

$\psi(r)$ 은 거리 r 에 따른 수소원자의 바닥상태 파동함수이다. 위 적분은 수소원자에서 전자가 핵으로부터 $\frac{a_0}{2}$ 보다 멀리서 발견될 확률이 약 91.97%임을 의미한다. (a_0 는 보어 반지름이다.) [2021 물리인증 1급 기출]

$$\begin{aligned} 17. W_n &:= \int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \sin^{n-1} x \cdot \sin x dx \\ &= [-\cos x \cdot \sin^{n-1} x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (n-1) \sin^{n-2} x \cos x \cdot (-\cos x) dx \\ &= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x dx = (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x (1 - \sin^2 x) dx \\ &= (n-1)(W_{n-2} - W_n) \end{aligned}$$

$$n W_n = (n-1) W_{n-2}, \quad W_n = \frac{n-1}{n} W_{n-2} \quad (n \geq 2)$$

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$$W_0 = \int_0^{\frac{\pi}{2}} 1 dx = \frac{\pi}{2}, \quad W_1 = \int_0^{\frac{\pi}{2}} \sin x dx = 1$$

이므로

$$W_n = \frac{n-1}{n} W_{n-2} = \frac{(n-1)(n-3)}{n(n-2)} W_{n-4} = \dots$$

$$= \begin{cases} \frac{(n-1)(n-3)\cdots 1}{n(n-2)\cdots 2} \cdot W_0 & (n=2m) \\ \frac{(n-1)(n-3)\cdots 2}{n(n-2)\cdots 1} \cdot W_1 & (n=2m+1) \end{cases}$$

$$\therefore W_n = \frac{(n-1)!!}{n!!} \cdot \left(\frac{\pi}{2}\right)^{\frac{1+(-1)^n}{2}} \quad (n > 0), \quad W_0 = \frac{\pi}{2} \quad (\text{Wallis' Formula})$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^{10} x dx = W_{10} = \frac{9 \cdot 7 \cdot 5 \cdot 3 \cdot 1}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{63\pi}{512} \quad \blacksquare$$

18. $I(a) := \int_0^1 \frac{x^a - 1}{\ln x} dx$, 파인만 적분 테크닉을 적용하면

$$\frac{d}{da} I(a) = \int_0^1 \frac{\partial}{\partial a} \left(\frac{x^a - 1}{\ln x} \right) dx = \int_0^1 x^a dx = \left[\frac{1}{a+1} x^{a+1} \right]_0^1 = \frac{1}{a+1}$$

$$I(a) = \int \frac{1}{a+1} da = \ln|a+1| + C$$

이때 $I(0) = \int_0^1 0 dx = 0$ 이므로 $C = 0$ 이다.

$$\therefore I(2022) = \int_0^1 \frac{x^{2022} - 1}{\ln x} dx = \ln 2023 \quad \blacksquare$$

19. $(1+x)^{-1}$ 의 이항급수를 구하고 양변을 적분하면

$$(1+x)^{-1} = \sum_{n=0}^{\infty} \binom{-1}{n} \cdot x^n = \sum_{n=0}^{\infty} \prod_{i=0}^{n-1} (-1-i) \cdot \frac{x^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \cdot x^n \quad (|x| < 1)$$

$$\int (1+x)^{-1} dx = \ln|1+x| = \int \sum_{n=0}^{\infty} (-1)^n x^n dx + C = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + C$$

$x=0$ 을 대입하면 $0 = 0 + C$ 에서 $C=0$ 이다.

$$\begin{aligned}
\therefore \int_0^1 \frac{\ln x}{x-1} dx &= \int_{-1}^0 \frac{\ln(t+1)}{t} dt = \int_{-1}^0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} t^n dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} t^{n+1} \right]_{-1}^0 \\
&= \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} 0^{n+1} \right] - \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} (-1)^{n+1} \right] = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} \\
&= \zeta(2) = \frac{\pi^2}{6} \quad (\because \text{바젤 문제에 의해 } \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6})
\end{aligned}$$

$$\begin{aligned}
20. \quad I(\alpha) &:= \int_0^{\infty} \frac{\sin x}{x} e^{-\alpha x} dx, \quad \frac{d}{d\alpha} I(\alpha) = \int_0^{\infty} \frac{\partial}{\partial \alpha} \frac{\sin x}{x} e^{-\alpha x} dx = - \int_0^{\infty} e^{-\alpha x} \sin x dx \\
&= - \left[-e^{-\alpha x} \cos x \right]_0^{\infty} + \int_0^{\infty} \alpha e^{-\alpha x} \cos x dx = -1 + \left[\alpha e^{-\alpha x} \sin x \right]_0^{\infty} + \int_0^{\infty} \alpha^2 e^{-\alpha x} \sin x dx \\
&= -1 - \alpha^2 I'(\alpha), \quad I'(\alpha) = - \frac{1}{1+\alpha^2}
\end{aligned}$$

$$I(\alpha) = \int I'(\alpha) d\alpha = C - \tan^{-1} \alpha \quad (C \in \mathbb{R}), \quad 0 = \lim_{\alpha \rightarrow \infty} I(\alpha) = C - \frac{\pi}{2} \text{에서 } C = \frac{\pi}{2} \text{이다.}$$

$$\therefore \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = 2 \int_0^{\infty} \frac{\sin x}{x} dx = 2I(0) = \pi \quad \blacksquare$$

$$\begin{aligned}
21. \quad \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx &= \left[-\frac{\sin^2 x}{x} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{2 \sin x \cos x}{x} dx = \int_{-\infty}^{\infty} \frac{\sin 2x}{x} dx \\
&= \int_{-\infty}^{\infty} \frac{\sin t}{t} dt \quad (t = 2x, \quad dt = 2dx)
\end{aligned}$$

$$\begin{aligned}
I(\alpha) &:= \int_0^{\infty} \frac{\sin x}{x} e^{-\alpha x} dx, \quad \frac{d}{d\alpha} I(\alpha) = \int_0^{\infty} \frac{\partial}{\partial \alpha} \frac{\sin x}{x} e^{-\alpha x} dx = - \int_0^{\infty} e^{-\alpha x} \sin x dx \\
&= - \left[-e^{-\alpha x} \cos x \right]_0^{\infty} + \int_0^{\infty} \alpha e^{-\alpha x} \cos x dx = -1 + \left[\alpha e^{-\alpha x} \sin x \right]_0^{\infty} + \int_0^{\infty} \alpha^2 e^{-\alpha x} \sin x dx \\
&= -1 - \alpha^2 I'(\alpha), \quad I'(\alpha) = - \frac{1}{1+\alpha^2}
\end{aligned}$$

$$I(\alpha) = \int I'(\alpha) d\alpha = C - \tan^{-1} \alpha \quad (C \in \mathbb{R}), \quad 0 = \lim_{\alpha \rightarrow \infty} I(\alpha) = C - \frac{\pi}{2} \text{에서 } C = \frac{\pi}{2} \text{이다.}$$

$$\therefore \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = 2 \int_0^{\infty} \frac{\sin x}{x} dx = 2I(0) = \pi \blacksquare$$

$$22. \quad I = \int_0^1 \sqrt[3]{2x^3 - 3x^2 - x + 1} dx = \int_0^1 \sqrt[3]{2(1-x)^3 - 3(1-x)^2 - (1-x) + 1} dx \quad (x \mapsto 1-x)$$

$$= \int_0^1 -\sqrt[3]{2x^3 - 3x^2 - x + 1} dx = -I$$

$$\therefore I = \int_0^1 \sqrt[3]{2x^3 - 3x^2 - x + 1} dx = 0 \blacksquare$$

$$23. \quad x = \tan t, \quad dx = \sec^2 t dt, \quad \frac{dx}{1+x^2} = dt$$

$$I = \int_0^1 \frac{\ln(x+1)}{x^2+1} dx = \int_0^{\frac{\pi}{4}} \ln(1+\tan t) dt = \int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4}-t\right)\right) dt \quad (t \mapsto \frac{\pi}{4}-t)$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan t}{1+\tan t}\right) dt = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan t}\right) dt = \int_0^{\frac{\pi}{4}} \{\ln 2 - \ln(1+\tan t)\} dt$$

$$2I = \int_0^{\frac{\pi}{4}} \{\ln(1+\tan t) + (\ln 2 - \ln(1+\tan t))\} dt = \int_0^{\frac{\pi}{4}} \ln 2 dt = \frac{\pi}{4} \ln 2$$

$$\therefore I = \int_0^1 \frac{\ln(x+1)}{x^2+1} dx = \frac{\pi}{8} \ln 2 \blacksquare$$

$$24. \quad I = \int_0^2 \frac{\ln(x+1)}{x^2-x+1} dx = \int_1^3 \frac{\ln t}{t^2-3t+3} dt \quad (t = x+1, \quad dt = dx)$$

$$= \int_3^1 \frac{\ln(3u^{-1})}{9u^{-2}-9u^{-1}+3} \left(-\frac{3}{u^2} du\right) \quad \left(u = \frac{3}{t}, \quad du = -\frac{3}{t^2} dt, \quad dt = -\frac{3}{u^2} du\right)$$

$$= \int_1^3 \frac{\ln 3 - \ln u}{u^2-3u+3} du = \int_1^3 \frac{\ln 3}{u^2-3u+3} du - \int_1^3 \frac{\ln u}{u^2-3u+3} du = \int_1^3 \frac{\ln 3}{u^2-3u+3} du - I$$

$$\begin{aligned}
&= \frac{\ln 3}{2} \int_1^3 \frac{1}{u^2 - 3u + 3} du = \frac{\ln 3}{2} \int_1^3 \frac{1}{\left(u - \frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} du = \frac{\ln 3}{2} \left[\frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2x-3}{\sqrt{3}} \right) \right]_1^3 \\
&= \frac{\ln 3}{\sqrt{3}} \left(\frac{\pi}{3} + \frac{\pi}{6} \right) = \frac{\pi \ln 3}{2\sqrt{3}} \blacksquare
\end{aligned}$$

$$\begin{aligned}
25. \quad I(\alpha) &:= \int_0^1 \frac{\ln(\alpha x^2 + 1)}{x+1} dx, \quad \frac{d}{d\alpha} I(\alpha) = \int_0^1 \frac{\partial}{\partial \alpha} \frac{\ln(\alpha x^2 + 1)}{x+1} dx = \int_0^1 \frac{x^2}{(\alpha x^2 + 1)(x+1)} dx \\
&= \int_0^1 \left(\frac{x-1}{(\alpha+1)(\alpha x^2 + 1)} + \frac{1}{(\alpha+1)(x+1)} \right) dx = \frac{1}{\alpha+1} \int_0^1 \left(\frac{x}{\alpha x^2 + 1} - \frac{1}{\alpha x^2 + 1} + \frac{1}{x+1} \right) dx \\
&= \frac{1}{\alpha+1} \left[\frac{1}{2\alpha} \ln |\alpha x^2 + 1| - \frac{1}{\sqrt{\alpha}} \tan^{-1}(\sqrt{\alpha} x) + \ln |x+1| \right]_0^1 \\
&= \frac{1}{\alpha+1} \left(\frac{\ln(\alpha+1)}{2\alpha} - \frac{\tan^{-1}(\sqrt{\alpha})}{\sqrt{\alpha}} + \ln 2 \right) \\
\therefore \int_0^1 \frac{\ln(x^2 + 1)}{x+1} dx &= I(1) = \frac{1}{2} \int_0^1 \frac{\ln(\alpha+1)}{\alpha(\alpha+1)} d\alpha - \int_0^1 \frac{\tan^{-1}(\alpha)}{\sqrt{\alpha}(\alpha+1)} d\alpha + \int_0^1 \frac{\ln 2}{\alpha+1} d\alpha \\
&= \frac{1}{2} \left(\int_0^1 \frac{\ln(\alpha+1)}{\alpha} d\alpha - \int_0^1 \frac{\ln(\alpha+1)}{\alpha+1} d\alpha \right) - 2 \left[\frac{1}{2} (\tan^{-1}(\alpha))^2 \right]_0^1 + \ln 2 [\ln(\alpha+1)]_0^1 \\
&= \frac{1}{2} \left(\int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^n}{n+1} d\alpha - \left[\frac{1}{2} (\ln(\alpha+1))^2 \right]_0^1 \right) - \frac{\pi^2}{16} + (\ln 2)^2 \\
&= \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \int_0^1 \alpha^n d\alpha - \frac{1}{2} (\ln 2)^2 \right) - \frac{\pi^2}{16} + (\ln 2)^2 \\
&= \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} \right) - \frac{\pi^2}{16} + \frac{3}{4} (\ln 2)^2 = \frac{1}{2} \left(\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{2n^2} \right) - \frac{\pi^2}{16} + \frac{3}{4} (\ln 2)^2 \\
&= \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{\pi^2}{16} + \frac{3}{4} (\ln 2)^2 = \frac{1}{4} \zeta(2) - \frac{\pi^2}{16} + \frac{3}{4} (\ln 2)^2 \\
&= \frac{3}{4} (\ln 2)^2 - \frac{\pi^2}{48} \blacksquare \quad (\because \text{바젤 문제에 의해 } \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6})
\end{aligned}$$

$$\begin{aligned}
26. \quad I &= \int_0^{\frac{\pi}{2}} \ln(\cos x) dx = \int_0^{\frac{\pi}{2}} \ln(\sin x) dx \quad (x \mapsto \frac{\pi}{2} - x) \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln(\sin x \cos x) dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{2} \sin 2x\right) dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln(\sin 2x) dx - \frac{\pi}{4} \ln 2 \\
&= \frac{1}{4} \int_0^{\pi} \ln(\sin t) dt - \frac{\pi}{4} \ln 2 \quad (t = 2x, \quad dt = 2dx) \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln(\sin t) dt - \frac{\pi}{4} \ln 2 = \frac{1}{2} I - \frac{\pi}{4} \ln 2, \quad \frac{1}{2} I = -\frac{\pi}{4} \ln 2 \\
\therefore I &= \int_0^{\frac{\pi}{2}} \ln(\cos x) dx = -\frac{\pi}{4} \ln 2 \quad \blacksquare
\end{aligned}$$

$$\begin{aligned}
27. \quad \text{sol 1) } I &= \int_0^1 \frac{\ln x}{x+1} dx = \int_0^1 \frac{\ln x}{1-x} dx - \int_0^1 \frac{2x \ln x}{1-x^2} dx \\
&= \int_0^1 \frac{\ln x}{1-x} dx - \frac{1}{2} \int_0^1 \frac{\ln t}{1-t} dt \quad (t = x^2, \quad dt = 2x dx) \\
&= \frac{1}{2} \int_0^1 \frac{\ln t}{1-t} dt
\end{aligned}$$

$(1+x)^{-1}$ 의 이항급수를 구하고 양변을 적분하면

$$(1+x)^{-1} = \sum_{n=0}^{\infty} \binom{-1}{n} \cdot x^n = \sum_{n=0}^{\infty} \prod_{i=0}^{n-1} (-1-i) \cdot \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot x^n \quad (|x| < 1)$$

$$\int (1+x)^{-1} dx = \ln|1+x| = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n dx + C = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + C$$

$x=0$ 을 대입하면 $0=0+C$ 에서 $C=0$ 이다.

$$\begin{aligned}
\int_0^1 \frac{\ln x}{x-1} dx &= \int_{-1}^0 \frac{\ln(t+1)}{t} dt = \int_{-1}^0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} t^n dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} t^{n+1} \right]_{-1}^0 \\
&= \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} 0^{n+1} \right] - \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} (-1)^{n+1} \right] = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2}
\end{aligned}$$

$$= \zeta(2) = \frac{\pi^2}{6} \quad (\because \text{바젤 문제에 의해 } \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6})$$

$$\therefore I = \frac{1}{2} \int_0^1 \frac{\ln t}{1-t} dt = -\frac{\pi^2}{12}$$

sol 2) $\int_0^1 x^a dx = \frac{1}{a+1}$ 의 양변을 a 로 미분하면

$$\frac{d}{da} \int_0^1 x^a dx = \int_0^1 \frac{\partial}{\partial a} x^a dx = \int_0^1 x^a \ln x dx = -\frac{1}{(a+1)^2}$$

한편 $|x| < 1$ 일 때 $\sum_{a=0}^{\infty} (-1)^a x^a = \frac{1}{1+x}$ 임을 이용하면

$$\sum_{a=0}^{\infty} (-1)^a \int_0^1 x^a \ln x dx = \int_0^1 \sum_{a=0}^{\infty} (-1)^a x^a \ln x dx = -\sum_{a=0}^{\infty} \frac{(-1)^a}{(a+1)^2}$$

$$\therefore \int_0^1 \frac{\ln x}{x+1} dx = -\sum_{a=0}^{\infty} \frac{(-1)^a}{(a+1)^2} = \sum_{a=1}^{\infty} \frac{(-1)^a}{a^2} = \sum_{a=1}^{\infty} \frac{1}{2a^2} - \sum_{a=1}^{\infty} \frac{1}{a^2}$$

$$= -\frac{1}{2} \sum_{a=1}^{\infty} \frac{1}{a^2} = -\frac{1}{2} \zeta(2) = -\frac{\pi^2}{12} \quad \blacksquare$$

28. $n \in \mathbb{N} \cup \{0\}$, $a_n := \int_0^{\frac{\pi}{2}} \cos^n x \cos nx dx$

$$a_n = \int_0^{\frac{\pi}{2}} \cos^n x \{ \cos(n+1)x \cos x + \sin(n+1)x \sin x \} dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n+1} x \cos(n+1)x dx + \int_0^{\frac{\pi}{2}} \cos^n x \sin(n+1)x \sin x dx$$

$$= a_{n+1} + \left[\sin(n+1)x \cdot \frac{-1}{n+1} \cos^{n+1} x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \cos(n+1)x (-\cos^{n+1} x) dx$$

$$= 2a_{n+1}$$

$$a_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}, \quad a_1 = \int_0^{\frac{\pi}{2}} \cos^2 dx = \left[\frac{1}{2}x + \frac{1}{4} \sin 2x \right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

$$a_n = \int_0^{\frac{\pi}{2}} \cos^n x \cos nx dx = \pi \cdot \left(\frac{1}{2} \right)^{n+1}$$

$$\therefore a_{2022} = \int_0^{\frac{\pi}{2}} \cos^{2022} x \cos 2022x dx = \frac{\pi}{2^{2023}} \blacksquare$$

29. sol 1) $t = -\ln x$, $x = e^{-t}$, $dx = -e^{-t} dt$

$$\int_0^1 (\ln x)^{2022} dx = \int_0^{\infty} t^{2022} e^{-t} dt = \Gamma(2023) = 2022! \blacksquare$$

$$\text{sol 2) } J_n := \int_0^1 (-\ln x)^n dx = [x \cdot (-\ln x)^n]_0^1 - \int_0^1 x \cdot n(-\ln x)^{n-1} \cdot \frac{-1}{x} dx$$

$$= 0 - \lim_{x \rightarrow 0} \{x \cdot (-\ln x)^n\} + n \int_0^1 (-\ln x)^{n-1} dx = nJ(n-1)$$

$$(\because x = e^{-t}, \lim_{x \rightarrow 0} \{x \cdot (-\ln x)^n\} = \lim_{t \rightarrow \infty} \frac{t^n}{e^t} = 0 \text{ by l'Hospital's rule})$$

$$J_n = nJ_{n-1} = n(n-1)J_{n-2} = \dots = n(n-1) \dots 1 \cdot J_0 = n!$$

$$\therefore \int_0^1 (\ln x)^{2022} dx = J_{2022} = 2022! \blacksquare$$

$$30. \int_1^e \frac{\ln x}{\sqrt{x}} dx = [\ln x \cdot 2\sqrt{x}]_1^e - \int_1^e \frac{2\sqrt{x}}{x} dx = 2\sqrt{e} - [4\sqrt{x}]_1^e = 4 - 2\sqrt{e} \blacksquare$$

$$31. I(\alpha) := \int_0^{\frac{\pi}{2}} \ln(a + \tan^2 x) dx, \quad \frac{d}{d\alpha} I(\alpha) = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial \alpha} \ln(a + \tan^2 x) dx = \int_0^{\frac{\pi}{2}} \frac{1}{a + \tan^2 x} dx$$

$$= \int_0^{\infty} \frac{1}{(t^2 + 1)(a + t^2)} dt = \frac{1}{a-1} \int_0^{\infty} \left(\frac{1}{t^2 + 1} - \frac{1}{t^2 + a} \right) dt \quad (t = \tan x, dt = \sec^2 x dx)$$

$$= \frac{1}{a-1} \left[\tan^{-1} t - \frac{1}{\sqrt{a}} \tan^{-1} \left(\frac{t}{\sqrt{a}} \right) \right]_0^{\infty} = \frac{\pi}{2} \left(\frac{1}{a-1} - \frac{1}{\sqrt{a}(a-1)} \right)$$

$$I(\alpha) = \int \frac{\pi}{2} \left(\frac{1}{a-1} - \frac{1}{\sqrt{a}(a-1)} \right) da = \frac{\pi}{2} \int \left(\frac{1}{a-1} - \frac{1}{\sqrt{a}(a-1)} \right) dx$$

$$= \frac{\pi}{2} \left(\ln(a-1) + \ln \left(\frac{\sqrt{a}+1}{\sqrt{a}-1} \right) \right) = \pi \ln(\sqrt{a}+1)$$

$$(\because I(0) = 2 \int_0^{\frac{\pi}{2}} \ln(\tan x) dx = 2 \int_0^{\frac{\pi}{2}} \ln(\cot x) dx = -I(0) = 0)$$

$$\therefore \int_0^{\frac{\pi}{2}} \ln(2 + \tan^2 x) dx = \pi \ln(\sqrt{2}+1) \blacksquare$$

$$32. I(\alpha) := \int_0^{\infty} \frac{\alpha x - \sin(\alpha x)}{x^3(x^2+1)} dx, \quad \frac{d^3}{d\alpha^3} I(\alpha) = \int_0^{\infty} \frac{\partial^3}{\partial \alpha^3} \frac{\alpha x - \sin(\alpha x)}{x^3(x^2+1)} dx$$

$$= \int_0^{\infty} \frac{\partial^2}{\partial \alpha^2} \frac{1 - \cos(\alpha x)}{x^2(x^2+1)} dx = \int_0^{\infty} \frac{\partial}{\partial \alpha} \frac{\sin(\alpha x)}{x(x^2+1)} dx = \int_0^{\infty} \frac{\cos(\alpha x)}{x^2+1} dx$$

$$J(\alpha) := \int_{-\infty}^{\infty} \frac{\cos \alpha x}{x^2+1} dx = \left[\frac{\sin(\alpha x)}{\alpha(x^2+1)} \right]_{-\infty}^{\infty} + \frac{2}{\alpha} \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2+1)^2} dx$$

$$= \frac{2}{\alpha} \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2+1)^2} dx, \quad \alpha J(\alpha) = 2 \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2+1)^2} dx$$

양변을 미분하면

$$\frac{d}{d\alpha} (\alpha J(\alpha)) = J(\alpha) + \alpha J'(\alpha) = 2 \int_{-\infty}^{\infty} \frac{\partial}{\partial \alpha} \frac{x \sin(\alpha x)}{(x^2+1)^2} dx = 2 \int_{-\infty}^{\infty} \frac{x^2 \cos(\alpha x)}{(x^2+1)^2} dx$$

$$= 2 \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{x^2+1} dx - 2 \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{(x^2+1)^2} dx = 2J(\alpha) - 2 \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{(x^2+1)^2} dx$$

$$\alpha J'(\alpha) - J(\alpha) = -2 \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{(x^2+1)^2} dx, \quad \text{다시 양변을 } \alpha \text{로 미분하면}$$

$$\alpha J''(\alpha) = -2 \int_{-\infty}^{\infty} \frac{\partial}{\partial \alpha} \frac{\cos(\alpha x)}{(x^2+1)^2} dx = 2 \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2+1)^2} dx = \alpha J(\alpha)$$

$J''(\alpha) = J(\alpha)$ 라는 미분방정식을 얻는다.

$J(\alpha)$ 의 이계도함수가 $J(\alpha)$ 와 동일해야 하므로 $J(\alpha)$ 는 지수함수 또는 삼각함수들의 합으로 구성되어 있어야 하고, 상수 C_1, C_2, C_3, C_4 에 대하여

$J(\alpha) = C_1 e^{f(\alpha)} + C_2 e^{-f(\alpha)}$ 와 $J(\alpha) = C_3 \sin g(\alpha) + C_4 \cos g(\alpha)$ 의 형태를 모두 시험해보면 다음과 같은 해를 얻는다. (오일러공식 $e^{ix} = \cos x + i \sin x$ 를 적용하면 두 경우가 동일함을 알 수 있다.)

$$J(\alpha) = C_1 e^\alpha + C_2 e^{-\alpha} \quad (\alpha > 0)$$

$$\text{한편 } J(0) = C_1 + C_2 = \int_{-\infty}^{\infty} \frac{1}{x^2 + 1} dx = [\tan^{-1} x]_{-\infty}^{\infty} = \pi,$$

$$J(\alpha) = \frac{2}{\alpha} \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2 + 1)^2} dx \text{에서 } \lim_{\alpha \rightarrow \infty} J(\alpha) = 0, \quad C_1 = 0 \text{이고 } C_2 = \pi \text{이다.}$$

$$\therefore J(\alpha) = \frac{\pi}{e^\alpha}$$

$$\frac{d^3}{d\alpha^3} I(\alpha) = \int_0^\infty \frac{\cos(\alpha x)}{x^2 + 1} dx = \frac{1}{2} J(\alpha) = \frac{\pi}{2e^\alpha} \text{이고}$$

$$I(0) = I'(0) = I''(0) = 0 \text{이므로}$$

$$I''(\alpha) = \frac{\pi}{2}(1 - e^{-\alpha}), \quad I'(\alpha) = \frac{\pi}{2}(\alpha + e^{-\alpha} - 1), \quad I(\alpha) = \frac{\pi}{2} \left(\frac{1}{2} \alpha^2 - e^{-\alpha} - \alpha + 1 \right) \text{이다.}$$

한편 $x = 2t, dx = 2dt$ 의 치환을 하면

$$\int_0^\infty \frac{x - \sin x}{x^3(x^2 + 4)} dx = \int_0^\infty \frac{2t - \sin(2t)}{8t^3(4t^2 + 4)} \cdot 2dt = \frac{1}{16} \int_0^\infty \frac{2t - \sin(2t)}{t^3(t^2 + 1)} dt = \frac{1}{16} I(2)$$

$$= \frac{\pi}{32}(1 - e^{-2}) \blacksquare$$

33. $u = \tan \frac{x}{2}, du = \frac{1}{2} \sec^2 \frac{x}{2} dx$

$$\int_0^\pi \frac{1 - \sin x}{1 + \sin x} dx = \int_0^\infty \frac{2 \left(1 - \frac{2u}{u^2 + 1} \right)}{(u^2 + 1) \left(\frac{2u}{u^2 + 1} + 1 \right)} du = \int_0^\infty \frac{2(u-1)^2}{(u^2 + 1)(u+1)^2} du$$

$$= 2 \int_0^{\infty} \left(\frac{2}{(u+1)^2} - \frac{1}{u^2+1} \right) du = 2 \left[-\frac{2}{u+1} - \tan^{-1}u \right]_0^{\infty} = 4 - \pi \blacksquare$$

$$\begin{aligned} 34. \int_0^{\frac{3}{5}} \frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}} dx &= \int_0^{\frac{3}{5}} \left(\frac{e^x}{(1-x)\sqrt{1-x^2}} + e^x \sqrt{\frac{1+x}{1-x}} \right) dx \\ &= \int_0^{\frac{3}{5}} \left(e^x \left(\sqrt{\frac{1+x}{1-x}} \right)' + (e^x)' \sqrt{\frac{1+x}{1-x}} \right) dx = \left[e^x \sqrt{\frac{1+x}{1-x}} \right]_0^{\frac{3}{5}} = 2e^{\frac{3}{5}} - 1 \blacksquare \end{aligned}$$

$$\begin{aligned} 35. \int_0^{\frac{\pi}{2}} \sin 2022x \cdot \sin^{2020} x dx &= \int_0^{\frac{\pi}{2}} \sin(2021x+x) \cdot \sin^{2020} x dx \\ &= \int_0^{\frac{\pi}{2}} \sin 2021x \cdot \cos x \cdot \sin^{2020} x dx + \int_0^{\frac{\pi}{2}} \cos 2021x \cdot \sin^{2021} x dx \\ &= \sin 2021x \cdot \frac{1}{2021} \sin^{2021} x - \frac{2021}{2021} \int_0^{\frac{\pi}{2}} \cos 2021x \cdot \sin^{2021} x dx + \int_0^{\frac{\pi}{2}} \cos 2021x \cdot \sin^{2021} x dx \\ &= \left[\frac{1}{2021} \sin 2021x \cdot \sin^{2021} x \right]_0^{\frac{\pi}{2}} = \frac{1}{2021} \blacksquare \end{aligned}$$

$$36. t = x^2 + \frac{1}{x^2} - 1, \quad dt = 2 \left(x - \frac{1}{x^3} \right) dx$$

$$\begin{aligned} \int_1^{\sqrt{2}} \frac{x^4-1}{x^2\sqrt{x^4-x^2+1}} dx &= \int_1^{\sqrt{2}} \frac{x^4-1}{x^3\sqrt{x^2+\frac{1}{x^2}-1}} dx = \int_0^{\sqrt{2}} \frac{x-1/x^3}{\sqrt{x^2+\frac{1}{x^2}-1}} dx \\ &= \frac{1}{2} \int_1^{\frac{3}{2}} \frac{1}{\sqrt{t}} dt = \left[\sqrt{t} \right]_1^{\frac{3}{2}} = \frac{\sqrt{6}}{2} - 1 \blacksquare \end{aligned}$$

$$37. u = \cos x, \quad du = -\sin x dx$$

$$\begin{aligned} \int_0^{\frac{\pi}{3}} \frac{\sin 2x}{2+\cos x} dx &= (-2) \int_0^{\frac{\pi}{3}} \frac{\cos x(-\sin x)}{2+\cos x} dx = (-2) \int_1^{\frac{1}{2}} \frac{u}{2+u} du = 2 \int_{\frac{1}{2}}^1 \left(1 - \frac{2}{u+2} \right) du \\ &= 2 \left[u - 2 \ln |u+2| \right]_{\frac{1}{2}}^1 = 2(1 - \ln 9) - 2 \left(\frac{1}{2} - 2 \ln \frac{5}{2} \right) = 1 - \ln \left(\frac{1296}{625} \right) \blacksquare \end{aligned}$$

38. (19)의 reduction formula를 이용하면

$$\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx \quad \text{에서}$$

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \tan^9 x dx &= \left[\frac{\tan^8 x}{8} \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^7 x dx = \frac{1}{8} - \left[\frac{\tan^6 x}{6} \right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \tan^5 x dx \\ &= \frac{1}{8} - \frac{1}{6} + \left[\frac{\tan^4 x}{4} \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^3 x dx = \frac{1}{8} - \frac{1}{6} + \frac{1}{4} - \left[\frac{\tan^2 x}{2} \right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \tan x dx \\ &= \frac{1}{8} - \frac{1}{6} + \frac{1}{4} - \frac{1}{2} + [-\ln|\cos x|]_0^{\frac{\pi}{4}} = \frac{1}{8} - \frac{1}{6} + \frac{1}{4} - \frac{1}{2} + \ln \sqrt{2} = \frac{1}{2} \ln 2 - \frac{7}{24} \quad \blacksquare \end{aligned}$$

$$39. I = \int_0^1 \frac{x-1}{(1+x^3)\ln x} dx = \frac{1}{2} \int_0^\infty \frac{x-1}{(1+x^3)\ln x} dx \quad (x \mapsto \frac{1}{x})$$

$$J(a) := \int_0^\infty \frac{x^a - 1}{(1+x^3)\ln x} dx, \quad J'(a) = \frac{\partial}{\partial a} J(a) = \int_0^\infty \frac{x^a}{1+x^3} dx = \frac{\pi}{3} \operatorname{csc}\left(\frac{\pi(a+1)}{3}\right)$$

$$\therefore I = \frac{1}{2} J(1) = \frac{1}{2} \int_0^1 J'(a) da = \frac{\pi}{6} \int_0^1 \operatorname{csc}\left(\frac{\pi(a+1)}{3}\right) da = \frac{\ln 3}{2} \quad \blacksquare$$

$$40. \text{ sol 1) } u = \frac{\sin t}{\sqrt{\cos t}}, \quad du = \frac{1 + \cos^2 t}{2\cos^{3/2} t} dt$$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{(1 + \sec^2 t) \sqrt{\sec t}}{(1 + \sec t)^2 - 2} dt &= \int_0^{\frac{\pi}{2}} \frac{1 + \cos^2 t}{\sqrt{\cos t} \{(\cos t + 1)^2 - 2\cos^2 t\}} dt \\ &= \int_0^{\frac{\pi}{2}} \frac{1 + \cos^2 t}{\sqrt{\cos t} (1 + 2\cos t - \cos^2 t)} dt = 2 \int_0^{\frac{\pi}{2}} \frac{\cos t}{1 + 2\cos t - \cos^2 t} \cdot \frac{1 + \cos^2 t}{2\cos^{3/2} t} dt \\ &= 2 \int_0^{\frac{\pi}{2}} \frac{1}{\frac{1}{\cos t} + 2 - \cos t} \cdot \frac{1 + \cos^2 t}{2\cos^{3/2} t} dt = 2 \int_0^{\frac{\pi}{2}} \frac{1}{\frac{1 - \cos^2 t}{\cos t} + 2} \cdot \frac{1 + \cos^2 t}{2\cos^{3/2} t} dt \\ &= 2 \int_0^{\frac{\pi}{2}} \frac{1}{\frac{\sin^2 t}{\cos t} + 2} \cdot \frac{1 + \cos^2 t}{2\cos^{3/2} t} dt = 2 \int_0^\infty \frac{1}{u^2 + 2} du = 2 \left[\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x}{\sqrt{2}} \right) \right]_0^\infty = \frac{\pi}{\sqrt{2}} \quad \blacksquare \end{aligned}$$

sol 2) $\text{sect} = x^2$, $\text{sectant} dt = 2x dx$, $u = \sqrt{x^2 - \frac{1}{x^2}}$, $du = \frac{x^4 + 1}{x^2 \sqrt{x^4 - 1}}$

$$I = \int_0^{\frac{\pi}{2}} \frac{(1 + \sec^2 t) \sqrt{\text{sect}}}{(1 + \text{sect})^2 - 2} dt = 2 \int_1^{\infty} \frac{x^4 + 1}{\sqrt{x^4 - 1} (x^4 + 2x^2 - 1)} dx = 2 \int_0^{\infty} \frac{1}{u^2 + 2} du$$

$$= 2 \left[\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x}{\sqrt{2}} \right) \right]_0^{\infty} = \frac{\pi}{\sqrt{2}} \blacksquare$$

41. $\int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - \sin 2x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - 2 \sin x \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \cos^2 x - 2 \sin x \cos x + \sin^2 x} dx$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + (\cos x - \sin x)^2} dx = \frac{1}{2} \left[\int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + (\cos x - \sin x)^2} dx + \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + (\cos x - \sin x)^2} dx \right]$$

($\because x \mapsto \frac{\pi}{2} - x : \int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + (\cos x - \sin x)^2} dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + (\cos x - \sin x)^2} dx$)

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{1 + (\cos x - \sin x)^2} dx = -\frac{1}{2} [\tan^{-1}(\cos x - \sin x)]_0^{\frac{\pi}{2}} = \frac{\pi}{4} \blacksquare$$

42. $\int_0^{\infty} (\ln(e^x + 1) - x) dx = \int_0^{\infty} \ln \left(\frac{e^x + 1}{e^x} \right) dx = \int_0^{\infty} \ln(1 + e^{-x}) dx$

$$= \int_0^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} e^{-nx}}{n} dx = \sum_{n=1}^{\infty} \int_0^{\infty} \frac{(-1)^{n-1} e^{-nx}}{n} dx = \sum_{n=1}^{\infty} \left[\frac{(-1)^n e^{-nx}}{n^2} \right]_0^{\infty}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} - 2 \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\zeta(2)}{2} = \frac{\pi^2}{12} \blacksquare$$

43. $I = \int_0^{\infty} \frac{x}{e^x + 1} dx = \int_1^{\infty} \frac{\ln t}{t(1+t)} dt$ ($x = \ln t$, $dx = \frac{1}{t} dt$)

$$= - \int_0^1 \frac{\ln u}{1+u} du \quad (u = \frac{1}{t}, \quad du = -\frac{1}{u^2} du)$$

$$= \int_0^1 \frac{\ln u}{1-u} du - \int_0^1 \frac{2u \ln u}{1-u^2} du = \int_0^1 \frac{\ln u}{1-u} du - \frac{1}{2} \int_0^1 \frac{\ln y}{1-y} dy \quad (y = u^2, \quad dy = 2u du)$$

$$= \frac{1}{2} \int_0^1 \frac{\ln u}{1-u} du$$

$(1+x)^{-1}$ 의 이항급수를 구하고 양변을 적분하면

$$(1+x)^{-1} = \sum_{n=0}^{\infty} \binom{-1}{n} \cdot x^n = \sum_{n=0}^{\infty} \prod_{i=0}^{n-1} (-1-i) \cdot \frac{x^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \cdot x^n \quad (|x| < 1)$$

$$\int (1+x)^{-1} dx = \ln|1+x| = \int \sum_{n=0}^{\infty} (-1)^n x^n dx + C = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + C$$

$x=0$ 을 대입하면 $0=0+C$ 에서 $C=0$ 이다.

$$\int_0^1 \frac{\ln x}{x-1} dx = \int_{-1}^0 \frac{\ln(t+1)}{t} dt = \int_{-1}^0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} t^n dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} t^{n+1} \right]_{-1}^0$$

$$= \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} 0^{n+1} \right] - \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} (-1)^{n+1} \right] = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$$

$$= \zeta(2) = \frac{\pi^2}{6} \quad (\because \text{바젤 문제에 의해 } \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6})$$

$$I = \int_0^{\infty} \frac{x}{e^x+1} dx = \int_1^{\infty} \frac{\ln t}{t(1+t)} dt = \frac{1}{2} \int_0^1 \frac{\ln u}{1-u} du = \frac{\pi^2}{12} \quad \blacksquare$$

$$44. \int_0^{\frac{\pi}{2}} \frac{\ln(\tan x)}{1-\tan x + \tan^2 x} dx = \int_0^{\infty} \frac{\ln t}{(1-t+t^2)(1+t^2)} dt \quad (t = \tan x, dt = \sec^2 x dx)$$

$$= \int_0^1 \frac{\ln t}{(1-t+t^2)(1+t^2)} dt + \int_1^{\infty} \frac{\ln t}{(1-t+t^2)(1+t^2)} dt$$

$$= \int_0^1 \frac{\ln t}{(1-t+t^2)(1+t^2)} dt - \int_0^1 \frac{u^2 \ln u}{(1-u+u^2)(1+u^2)} du \quad (u = \frac{1}{t}, du = -\frac{1}{t^2} dt)$$

$$= \int_0^1 \frac{(1-u^2) \ln u}{(1-u+u^2)(1+u^2)} du = \int_0^1 \frac{2u \ln u}{1+u^2} du - \int_0^1 \frac{(2u-1) \ln u}{1-u+u^2} du$$

$$= [\ln u \ln(u^2+1)]_0^1 - \int_0^1 \frac{1}{u} \ln(u^2+1) du - [\ln u \ln(1-u+u^2)]_0^1 + \int_0^1 \frac{1}{u} \ln(1-u+u^2) du$$

$$\begin{aligned}
&= -\int_0^1 \frac{\ln(1+u^2)}{u} du + \int_0^1 \frac{\ln(1-u+u^2)}{u} du \\
&= -\int_0^1 \frac{\ln(1+u^2)}{u^2} u du + \int_0^1 \frac{\ln(1+u^3)}{u} du - \int_0^1 \frac{\ln(1+u)}{u} du \\
&= -\frac{1}{2} \int_0^1 \frac{\ln(1+y)}{y} dy + \int_0^1 \frac{\ln(1+u^3)}{u^3} u^2 du - \int_0^1 \frac{\ln(1+u)}{u} du \quad (y=u^2, dy=2u du) \\
&= -\frac{3}{2} \int_0^1 \frac{\ln(1+u)}{u} du + \frac{1}{3} \int_0^1 \frac{\ln(1+z)}{z} dz \quad (z=u^3, dz=3u^2 du) \\
&= -\frac{7}{6} \int_0^1 \frac{\ln(1+u)}{u} du = -\frac{7}{6} \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n u^n}{n+1} du = -\frac{7}{6} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \int_0^1 u^n du \\
&= -\frac{7}{6} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} = -\frac{7}{6} \left(\sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \right) = -\frac{7}{12} \sum_{n=1}^{\infty} \frac{1}{n^2} \\
&= -\frac{7}{12} \zeta(2) = -\frac{7\pi^2}{72} \quad (\because \text{바젤 문제에 의해 } \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6})
\end{aligned}$$

45. $t = x - \frac{1}{x}, dt = \left(1 + \frac{1}{x^2}\right) dx, \frac{dx}{1+x^2} = \frac{x^2}{(1+x^2)^2} dt = \frac{dt}{(x+1/x)^2} = \frac{dt}{t^2+4}$

$$I = \int_{-\infty}^{\infty} \frac{\cos\left(x - \frac{1}{x}\right)}{1+x^2} dx = 2 \int_0^{\infty} \frac{\cos\left(x - \frac{1}{x}\right)}{1+x^2} dx = 2 \int_{-\infty}^{\infty} \frac{\cos t}{t^2+4} dt$$

$$J(\alpha) := \int_{-\infty}^{\infty} \frac{\cos \alpha x}{x^2+1} dx = \left[\frac{\sin(\alpha x)}{\alpha(x^2+1)} \right]_{-\infty}^{\infty} + \frac{2}{\alpha} \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2+1)^2} dx$$

$$= \frac{2}{\alpha} \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2+1)^2} dx, \quad \alpha J(\alpha) = 2 \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2+1)^2} dx$$

양변을 미분하면

$$\frac{d}{d\alpha}(\alpha J(\alpha)) = J(\alpha) + \alpha J'(\alpha) = 2 \int_{-\infty}^{\infty} \frac{\partial}{\partial \alpha} \frac{x \sin(\alpha x)}{(x^2+1)^2} dx = 2 \int_{-\infty}^{\infty} \frac{x^2 \cos(\alpha x)}{(x^2+1)^2} dx$$

$$= 2 \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{x^2+1} dx - 2 \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{(x^2+1)^2} dx = 2J(\alpha) - 2 \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{(x^2+1)^2} dx$$

$$\alpha J'(\alpha) - J(\alpha) = -2 \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{(x^2+1)^2} dx, \text{ 다시 양변을 } \alpha \text{로 미분하면}$$

$$\alpha J''(\alpha) = -2 \int_{-\infty}^{\infty} \frac{\partial}{\partial \alpha} \frac{\cos(\alpha x)}{(x^2+1)^2} dx = 2 \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2+1)^2} dx = \alpha J(\alpha)$$

$J''(\alpha) = J(\alpha)$ 라는 미분방정식을 얻는다.

$J(\alpha)$ 의 이계도함수가 $J(\alpha)$ 와 동일해야 하므로 $J(\alpha)$ 는 지수함수 또는 삼각함수들의 합으로 구성되어 있어야 하고, 상수 C_1, C_2, C_3, C_4 에 대하여

$J(\alpha) = C_1 e^{f(\alpha)} + C_2 e^{-f(\alpha)}$ 와 $J(\alpha) = C_3 \sin g(\alpha) + C_4 \cos g(\alpha)$ 의 형태를 모두 시험해보면 다음과 같은 해를 얻는다. (오일러공식 $e^{ix} = \cos x + i \sin x$ 를 적용하면 두 경우가 동일함을 알 수 있다.)

$$J(\alpha) = C_1 e^{\alpha} + C_2 e^{-\alpha} \quad (\alpha \geq 0)$$

$$\text{한편 } J(0) = C_1 + C_2 = \int_{-\infty}^{\infty} \frac{1}{x^2+1} dx = [\tan^{-1} x]_{-\infty}^{\infty} = \pi,$$

$$J(\alpha) = \frac{2}{\alpha} \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2+1)^2} dx \text{에서 } \lim_{\alpha \rightarrow \infty} J(\alpha) = 0, \quad C_1 = 0 \text{이고 } C_2 = \pi \text{이다.}$$

$$J(\alpha) = \int_{-\infty}^{\infty} \frac{\cos \alpha x}{x^2+1} dx = \frac{\pi}{e^{\alpha}}$$

$$\therefore I = 2 \int_{-\infty}^{\infty} \frac{\cos t}{t^2+4} dt = 4 \int_{-\infty}^{\infty} \frac{\cos 2u}{4u^2+4} du \quad (t = 2u, \quad dt = 2du)$$

$$= J(2) = \frac{\pi}{e^2} \quad \blacksquare$$

$$46. \quad I(\alpha) := \int_{-\infty}^{\infty} \frac{\cos \alpha x}{x^2+1} dx = \left[\frac{\sin(\alpha x)}{\alpha(x^2+1)} \right]_{-\infty}^{\infty} + \frac{2}{\alpha} \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2+1)^2} dx$$

$$= \frac{2}{\alpha} \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2+1)^2} dx, \quad \alpha I(\alpha) = 2 \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2+1)^2} dx$$

양변을 미분하면

$$\begin{aligned} \frac{d}{d\alpha}(\alpha I(\alpha)) &= I(\alpha) + \alpha I'(\alpha) = 2 \int_{-\infty}^{\infty} \frac{\partial}{\partial \alpha} \frac{x \sin(\alpha x)}{(x^2+1)^2} dx = 2 \int_{-\infty}^{\infty} \frac{x^2 \cos(\alpha x)}{(x^2+1)^2} dx \\ &= 2 \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{x^2+1} dx - 2 \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{(x^2+1)^2} dx = 2I(\alpha) - 2 \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{(x^2+1)^2} dx \end{aligned}$$

$$\alpha I'(\alpha) - I(\alpha) = -2 \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{(x^2+1)^2} dx, \text{ 다시 양변을 } \alpha \text{로 미분하면}$$

$$\alpha I''(\alpha) = -2 \int_{-\infty}^{\infty} \frac{\partial}{\partial \alpha} \frac{\cos(\alpha x)}{(x^2+1)^2} dx = 2 \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2+1)^2} dx = \alpha I(\alpha)$$

$I''(\alpha) = I(\alpha)$ 라는 미분방정식을 얻는다.

$I(\alpha)$ 의 이계도함수가 $I(\alpha)$ 와 동일해야 하므로 $I(\alpha)$ 는 지수함수 또는 삼각함수들의 합으로 구성되어 있어야 하고, 상수 C_1, C_2, C_3, C_4 에 대하여

$I(\alpha) = C_1 e^{f(\alpha)} + C_2 e^{-f(\alpha)}$ 와 $I(\alpha) = C_3 \sin g(\alpha) + C_4 \cos g(\alpha)$ 의 형태를 모두 시험해보면 다음과 같은 해를 얻는다. (오일러공식 $e^{ix} = \cos x + i \sin x$ 를 적용하면 두 경우가 동일함을 알 수 있다.)

$$I(\alpha) = C_1 e^\alpha + C_2 e^{-\alpha} \quad (\alpha \geq 0)$$

$$\text{한편 } I(0) = C_1 + C_2 = \int_{-\infty}^{\infty} \frac{1}{x^2+1} dx = [\tan^{-1} x]_{-\infty}^{\infty} = \pi,$$

$$I(\alpha) = \frac{2}{\alpha} \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2+1)^2} dx \text{에서 } \lim_{\alpha \rightarrow \infty} I(\alpha) = 0, \quad C_1 = 0 \text{이고 } C_2 = \pi \text{이다.}$$

$$I(\alpha) = \frac{\pi}{e^\alpha}, \quad \therefore \int_0^\infty \frac{\cos x}{x^2+1} dx = \frac{1}{2} I(1) = \frac{\pi}{2e} \quad \blacksquare$$

$$47. \quad I(\alpha) := \int_0^\infty \frac{x \sin(\alpha x)}{x^2+1} dx = \int_0^\infty \frac{x^2 \sin(\alpha x)}{x(x^2+1)} dx \quad (\alpha > 0)$$

$$= \int_0^\infty \frac{(1+x^2-1)\sin(\alpha x)}{x(x^2+1)} dx = \int_0^\infty \frac{\sin(\alpha x)}{x} dx - \int_0^\infty \frac{\sin(\alpha x)}{x(1+x^2)} dx$$

$$= \int_0^\infty \frac{\sin t}{t} dt - \int_0^\infty \frac{\sin(\alpha x)}{x(1+x^2)} dx \quad (t = \alpha x, \quad dt = \alpha dx)$$

$$= \frac{\pi}{2} - \int_0^{\infty} \frac{\sin(\alpha x)}{x(1+x^2)} dx \quad (\ast)$$

$$\frac{d}{d\alpha} I(\alpha) = - \int_0^{\infty} \frac{\partial}{\partial \alpha} \frac{\sin(\alpha x)}{x(1+x^2)} dx = - \int_0^{\infty} \frac{\cos(\alpha x)}{1+x^2} dx$$

$$\frac{d^2}{d\alpha^2} I(\alpha) = - \int_0^{\infty} \frac{\partial}{\partial \alpha} \frac{\cos(\alpha x)}{1+x^2} dx = \int_0^{\infty} \frac{x \sin(\alpha x)}{1+x^2} dx = I(\alpha)$$

$I''(\alpha) = I(\alpha)$ 라는 미분방정식을 얻는다.

$I(\alpha)$ 의 이계도함수가 $I(\alpha)$ 와 동일해야 하므로 $I(\alpha)$ 는 지수함수 또는 삼각함수들의 합으로 구성되어 있어야 하고, 상수 C_1, C_2, C_3, C_4 에 대하여

$I(\alpha) = C_1 e^{f(\alpha)} + C_2 e^{-f(\alpha)}$ 와 $I(\alpha) = C_3 \sin g(\alpha) + C_4 \cos g(\alpha)$ 의 형태를 모두 시험해보면 다음과 같은 해를 얻는다. (오일러공식 $e^{ix} = \cos x + i \sin x$ 를 적용하면 두 경우가 동일함을 알 수 있다.)

$$I(\alpha) = C_1 e^{\alpha} + C_2 e^{-\alpha} \quad (\alpha > 0)$$

한편

$$\lim_{\alpha \rightarrow 0^+} I(\alpha) = \lim_{\alpha \rightarrow 0^+} \left(\frac{\pi}{2} - \int_0^{\infty} \frac{\sin(\alpha x)}{x(1+x^2)} dx \right) = \frac{\pi}{2} \text{에서 } C_1 + C_2 = \frac{\pi}{2},$$

$$\lim_{\alpha \rightarrow 0^+} I'(\alpha) = \lim_{\alpha \rightarrow 0^+} \left(- \int_0^{\infty} \frac{\cos(\alpha x)}{1+x^2} dx \right) = -\frac{\pi}{2} \text{에서 } C_1 - C_2 = -\frac{\pi}{2} \text{를 얻는다.}$$

따라서 $C_1 = 0, C_2 = \frac{\pi}{2}$ 이고

$$I(\alpha) = \int_0^{\infty} \frac{x \sin(\alpha x)}{x^2 + 1} dx = \frac{\pi}{2e^{\alpha}},$$

$$\therefore \int_0^{\infty} \frac{x \sin x}{x^2 + 1} dx = I(1) = \frac{\pi}{2e} \quad \blacksquare$$

$$\ast J(\alpha) := \int_0^{\infty} \frac{\sin x}{x} e^{-\alpha x} dx, \quad \frac{d}{d\alpha} J(\alpha) = \int_0^{\infty} \frac{\partial}{\partial \alpha} \frac{\sin x}{x} e^{-\alpha x} dx = - \int_0^{\infty} e^{-\alpha x} \sin x dx$$

$$= - [-e^{-\alpha x} \cos x]_0^{\infty} + \int_0^{\infty} \alpha e^{-\alpha x} \cos x dx = -1 + [\alpha e^{-\alpha x} \sin x]_0^{\infty} + \int_0^{\infty} \alpha^2 e^{-\alpha x} \sin x dx$$

$$= -1 - \alpha^2 J'(\alpha), \quad J'(\alpha) = -\frac{1}{1+\alpha^2}$$

$$J(\alpha) = \int J'(\alpha) d\alpha = C - \tan^{-1} \alpha \quad (C \in \mathbb{R}), \quad 0 = \lim_{\alpha \rightarrow \infty} J(\alpha) = C - \frac{\pi}{2} \text{에서 } C = \frac{\pi}{2} \text{이다.}$$

$$\therefore \int_0^{\infty} \frac{\sin x}{x} dx = J(0) = \frac{\pi}{2} \quad \blacksquare$$

$$\begin{aligned} 48. \quad \int_0^{\infty} \frac{\ln(1+x)}{x(x^2+1)} dx &= \int_0^1 \frac{\ln(1+x)}{x(x^2+1)} dx + \int_1^{\infty} \frac{\ln(1+x)}{x(x^2+1)} dx \\ &= \int_0^1 \frac{\ln(1+x)}{x(x^2+1)} dx + \int_1^0 \frac{t \ln(1+t^{-1})}{t^{-2}+1} \left(-\frac{1}{t^2} dt\right) \quad \left(t = \frac{1}{x}, \quad dt = -\frac{1}{x^2} dx\right) \\ &= \int_0^1 \frac{\ln(1+x)}{x(x^2+1)} dx + \int_0^1 \frac{t \ln(1+t^{-1})}{1+t^2} dt \\ &= \int_0^1 \frac{\ln(1+x)}{x(x^2+1)} dx + \int_0^1 \frac{t \ln(1+t)}{1+t^2} dt - \int_0^1 \frac{t \ln t}{1+t^2} dt \\ &= \int_0^1 \left(t + \frac{1}{t}\right) \frac{\ln(1+t)}{1+t^2} dt - \int_0^1 \frac{t \ln t}{1+t^2} dt \\ &= \int_0^1 \frac{\ln(1+t)}{t} dt - \frac{1}{4} \int_0^1 \frac{\ln u}{1+u} du \quad (u = t^2, \quad du = 2t dt) \\ &= \int_0^1 \frac{\ln(1+t)}{t} dt - \frac{1}{4} \left([\ln t \ln(1+t)]_0^1 - \int_0^1 \frac{\ln(1+t)}{t} dt \right) = \frac{5}{4} \int_0^1 \frac{\ln(1+t)}{t} dt \\ &= \frac{5}{4} \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n u^n}{n+1} du = \frac{5}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \int_0^1 u^n du = \frac{5}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} \\ &= \frac{5}{4} \left(\sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \right) = \frac{5}{8} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{5}{8} \zeta(2) = \frac{5\pi^2}{48} \quad \blacksquare \end{aligned}$$

$$(\because \text{바젤 문제에 의해 } \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6})$$

※ $x = \tan t$ 의 치환을 한 후 $I(\alpha) := \int_0^{\frac{\pi}{2}} \frac{\ln(1 + \alpha \tan t)}{\tan t} dt$ 로 정의하면 파인만 테크닉을 적용하여

$$I(\alpha) = \frac{\pi + 2\alpha \ln \alpha}{2(1 + \alpha^2)}, \quad I(1) = \int_0^{\infty} \frac{\ln(1+x)}{x(1+x^2)} dx = \frac{5\pi^2}{48}$$

을 얻을 수 있다.

49. $v = \tan u, dv = \sec^2 u du$

$$\begin{aligned} I(\alpha) &:= \int_0^{\frac{\pi}{2}} \frac{\sin(u + \alpha \tan u)}{\sin u} du = \int_0^{\frac{\pi}{2}} \left(\cos(\alpha \tan u) + \frac{\sin(\alpha \tan u)}{\tan u} \right) du \\ &= \int_0^{\infty} \frac{\cos(\alpha v) + \frac{\sin(\alpha v)}{v}}{1+v^2} dv, \\ \frac{d}{d\alpha} I(\alpha) &= \int_0^{\infty} \frac{\partial}{\partial \alpha} \frac{\cos(\alpha v) + \frac{\sin(\alpha v)}{v}}{1+v^2} dv = \int_0^{\infty} \frac{\cos(\alpha v) - v \sin(\alpha v)}{1+v^2} dv \\ &= \int_0^{\infty} \frac{\cos(\alpha v)}{1+v^2} dv - \int_0^{\infty} \frac{v \sin(\alpha v)}{1+v^2} dv = \frac{\pi}{2e} - \frac{\pi}{2e} = 0 \quad (\because \#46, \#47) \\ \therefore I(\alpha) &= C = I(0) = \frac{\pi}{2} \\ \therefore \int_0^{\frac{\pi}{2}} \frac{\sin(u + \sqrt{\pi} \tan u)}{\sin u} du &= I(\sqrt{\pi}) = \frac{\pi}{2} \blacksquare \end{aligned}$$

$$\begin{aligned} 50. I(\alpha) &:= \int_0^{\alpha} \sin^2(x - \sqrt{\alpha^2 - x^2}) dx = \int_{\frac{\pi}{2}}^0 \sin^2(\alpha \cos t - \alpha \sin t) \cdot (-\alpha \sin t dt) \\ &= \int_0^{\frac{\pi}{2}} \sin^2(\alpha \cos t - \alpha \sin t) \alpha \sin t dt \quad (x = \alpha \cos t, dx = -\alpha \sin t dt) \\ I(\alpha) &:= \int_0^{\alpha} \sin^2(x - \sqrt{\alpha^2 - x^2}) dx = \int_0^{\frac{\pi}{2}} \sin^2(\alpha \sin t - \alpha \cos t) \alpha \cos t dt \end{aligned}$$

$$= \int_0^{\frac{\pi}{2}} \sin^2(\alpha \cos t - \alpha \sin t) \alpha \cos t dt \quad (x = \alpha \sin t, \quad dx = \alpha \cos t dt)$$

$$I(\alpha) = \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin^2(\alpha(\sin t - \cos t)) \alpha(\sin t + \cos t) dt$$

$$= \frac{1}{2} \int_{-\alpha}^{\alpha} \sin^2 u du \quad (u = \alpha(\sin t - \cos t), \quad du = \alpha(\cos t + \sin t) dt)$$

$$= \frac{1}{4} \int_{-\alpha}^{\alpha} (1 - \cos 2u) du = \frac{1}{4} \left[u - \frac{1}{2} \sin 2u \right]_{-\alpha}^{\alpha} = \frac{\alpha}{2} - \frac{1}{4} \sin 2\alpha$$

$$\therefore \int_0^{\pi} \sin^2(x - \sqrt{\pi^2 - x^2}) dx = I(\pi) = \frac{\pi}{2} \quad \blacksquare$$

51. $I(\alpha) := \int_0^{\infty} e^{-x^2} \cos(\alpha x) dx$

$$\frac{d}{d\alpha} I(\alpha) = \int_0^{\infty} \frac{\partial}{\partial \alpha} e^{-x^2} \cos(\alpha x) dx = - \int_0^{\infty} x e^{-x^2} \sin(\alpha x) dx$$

$$= - \left(\left[-\frac{1}{2} e^{-x^2} \sin(\alpha x) \right]_0^{\infty} + \frac{\alpha}{2} \int_0^{\infty} e^{-x^2} \cos(\alpha x) dx \right) = -\frac{\alpha}{2} I(\alpha)$$

$I'(\alpha) = -\frac{\alpha}{2} I(\alpha)$ 의 미분방정식을 얻는다. 이를 풀기 위해 변수를 분리하고 양변을 적분하면

$$\frac{1}{I(\alpha)} dI(\alpha) = -\frac{\alpha}{2} d\alpha, \quad \int \frac{1}{I(\alpha)} dI(\alpha) = -\int \frac{\alpha}{2} d\alpha, \quad \ln |I(\alpha)| + C = -\frac{1}{4} \alpha^2$$

$$I(0) = \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \quad (\because \text{가우스 적분}) \quad \text{이므로 } C = -\ln\left(\frac{\sqrt{\pi}}{2}\right)$$

$$I(\alpha) = \int_0^{\infty} e^{-x^2} \cos(\alpha x) dx = \frac{\sqrt{\pi}}{2} e^{-\frac{\alpha^2}{4}}$$

$$\therefore \int_0^{\infty} e^{-x^2} \cos 2x dx = I(2) = \frac{\sqrt{\pi}}{2e} \quad \blacksquare$$

$$52. I(\alpha) := \int_0^{\frac{\pi}{2}} \cos(\alpha \tan x) dx, \quad t = \tan x, \quad dt = \sec^2 x dx, \quad dx = \frac{dt}{1+t^2}$$

$$\begin{aligned} I(\alpha) &= \int_0^{\frac{\pi}{2}} \cos(\alpha \tan x) dx = \int_0^{\infty} \frac{\cos \alpha t}{1+t^2} dt = \frac{1}{\alpha} \left[\frac{\sin \alpha t}{1+t^2} \right]_0^{\infty} + \frac{1}{\alpha} \int_0^{\infty} \frac{2t \sin \alpha t}{(1+t^2)^2} dt \\ &= \frac{1}{\alpha} \int_0^{\infty} \frac{2t \sin \alpha t}{(1+t^2)^2} dt, \quad \alpha I(\alpha) \text{에 대하여 파인만 적분 테크닉을 이용하면} \end{aligned}$$

$$\begin{aligned} \frac{d}{d\alpha}(\alpha I(\alpha)) &= I(\alpha) + \alpha I'(\alpha) = \int_0^{\infty} \frac{\partial}{\partial \alpha} \left(\frac{2t \sin \alpha t}{(1+t^2)^2} \right) dt = \int_0^{\infty} \frac{2t^2 \cos \alpha t}{(1+t^2)^2} dt \\ &= \int_0^{\infty} \frac{2(t^2+1-1)\cos \alpha t}{(1+t^2)^2} dt = 2 \int_0^{\infty} \frac{\cos \alpha t}{1+t^2} dt - \int_0^{\infty} \frac{2\cos \alpha t}{(1+t^2)^2} dt \\ &= 2I(\alpha) - \int_0^{\infty} \frac{2\cos \alpha t}{(1+t^2)^2} dt. \end{aligned}$$

다시 양변을 α 에 대하여 미분하면

$$I'(\alpha) + \alpha I''(\alpha) + I'(\alpha) = 2I'(\alpha) - \int_0^{\infty} \frac{\partial}{\partial \alpha} \frac{2\cos \alpha t}{(1+t^2)^2} dt = 2I'(\alpha) + \int_0^{\infty} \frac{2t \sin \alpha t}{(1+t^2)^2} dt$$

$$\alpha I''(\alpha) = \int_0^{\infty} \frac{2t \sin \alpha t}{(1+t^2)^2} dt = \alpha I(\alpha), \quad I''(\alpha) = I(\alpha) \text{라는 미분방정식을 얻는다.}$$

$I(\alpha)$ 의 이계도함수가 $I(\alpha)$ 와 동일해야 하므로 $I(\alpha)$ 는 지수함수 또는 삼각함수들의 합으로 구성되어 있어야 하고, 상수 C_1, C_2, C_3, C_4 에 대하여

$I(\alpha) = C_1 e^{f(\alpha)} + C_2 e^{-f(\alpha)}$ 와 $I(\alpha) = C_3 \sin g(\alpha) + C_4 \cos g(\alpha)$ 의 형태를 모두 시험해보면 다음과 같은 해를 얻는다. (오일러공식 $e^{ix} = \cos x + i \sin x$ 를 적용하면 두 경우가 동일함을 알 수 있다.)

$$I(\alpha) = C_1 e^{\alpha} + C_2 e^{-\alpha} \quad (\alpha \geq 0)$$

이때 $I(\alpha)$ 는 $\cos(\alpha \tan x)$ 를 0부터 $\frac{\pi}{2}$ 까지 정적분한 값이므로 모든 $\alpha \geq 0$ 에 대하여 유계(bounded)이고, 따라서 $C_1 = 0$ 이다. 또한 $\alpha = 0$ 을 대입하면 $I(0) = \frac{\pi}{2} = C_2$ 를 얻는다.

$$I(\alpha) = \int_0^{\frac{\pi}{2}} \cos(\alpha \tan x) dx = \frac{\pi}{2e^\alpha},$$

$$\therefore \int_0^{\frac{\pi}{2}} \cos(\tan x) dx = I(1) = \frac{\pi}{2e} \blacksquare$$

$$53. I(\alpha) := \int_0^{\frac{\pi}{2}} \frac{\tan^{-1}(\alpha \tan x)}{\tan x} dx$$

$$\frac{d}{d\alpha} I(\alpha) = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial \alpha} \frac{\tan^{-1}(\alpha \tan x)}{\tan x} dx = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \alpha^2 \tan^2 x} dx$$

$$= \int_0^{\infty} \frac{1}{(1 + \alpha^2 t^2)(1 + t^2)} dt \quad (t = \tan x, \quad dt = \sec^2 x dx)$$

$$= \frac{1}{\alpha^2 - 1} \int_0^{\infty} \left(\frac{\alpha^2}{1 + \alpha^2 t^2} - \frac{1}{1 + t^2} \right) dt = \frac{1}{\alpha^2 - 1} [\alpha \tan^{-1}(\alpha t) - \tan^{-1} t]_0^{\infty} = \frac{\pi}{2(\alpha + 1)}$$

$$I(\alpha) = \frac{\pi}{2} \ln |\alpha + 1| + C, \quad C = I(0) = 0 \text{에서 } I(\alpha) = \frac{\pi}{2} \ln |\alpha + 1|$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{x}{\tan x} dx = I(1) = \frac{\pi}{2} \ln 2 \blacksquare$$

$$54. I(\alpha) := \int_0^{\pi} \ln(1 - 2\alpha \cos x + \alpha^2) dx, \quad \frac{d}{d\alpha} I(\alpha) = \int_0^{\pi} \frac{\partial}{\partial \alpha} \ln(1 - 2\alpha \cos x + \alpha^2) dx$$

$$= \int_0^{\pi} \frac{2(\alpha - \cos x)}{1 - 2\alpha \cos x + \alpha^2} dx = \frac{1}{\alpha} \int_0^{\pi} \frac{2(\alpha^2 - \alpha \cos x)}{1 - 2\alpha \cos x + \alpha^2} dx$$

$$= \frac{1}{\alpha} \int_0^{\pi} \left(\frac{2(\alpha^2 - \alpha \cos x)}{1 - 2\alpha \cos x + \alpha^2} - 1 + 1 \right) dx$$

$$= \frac{1}{\alpha} \int_0^{\pi} \left(\frac{2(\alpha^2 - \alpha \cos x)}{1 - 2\alpha \cos x + \alpha^2} - \frac{1 - 2\alpha \cos x + \alpha^2}{1 - 2\alpha \cos x + \alpha^2} + 1 \right) dx$$

$$= \frac{1}{\alpha} \int_0^{\pi} \left(1 - \frac{1 - \alpha^2}{1 + \alpha^2 - 2\alpha \cos x} \right) dx = \frac{\pi}{\alpha} - \frac{1 - \alpha^2}{\alpha(1 + \alpha^2)} \int_0^{\pi} \left(\frac{1}{1 - \frac{2\alpha}{1 + \alpha^2} \cos x} \right) dx$$

$$\begin{aligned}
&= \frac{\pi}{\alpha} - \frac{1-\alpha^2}{\alpha(1+\alpha^2)} \int_0^\infty \frac{2}{(1+t^2)\left(1-\frac{2\alpha}{1+\alpha^2}\frac{1-t^2}{1+t^2}\right)} dt \quad (t = \tan \frac{x}{2}, \quad dt = \frac{1}{2} \sec^2 \frac{x}{2} dx) \\
&= \frac{\pi}{\alpha} - \frac{1-\alpha^2}{\alpha(1+\alpha^2)} \int_0^\infty \frac{2}{\left(1-\frac{2\alpha}{1+\alpha^2}\right) + \left(1+\frac{2\alpha}{1+\alpha^2}\right)t^2} dt \\
&= \frac{\pi}{\alpha} - \frac{1-\alpha^2}{\alpha(1+\alpha^2)} \int_0^\infty \frac{2(1+\alpha^2)}{(1+\alpha^2-2\alpha) + (1+\alpha^2+2\alpha)t^2} dt \\
&= \frac{\pi}{\alpha} - \frac{1-\alpha^2}{\alpha(1+\alpha^2)} \int_0^\infty \frac{2(1+\alpha^2)}{(1-\alpha)^2 + (1+\alpha)^2 t^2} dt = \frac{\pi}{\alpha} - \frac{2(1+\alpha)}{\alpha(1-\alpha)} \int_0^\infty \frac{1}{1 + \left(\frac{\alpha+1}{\alpha-1}\right)^2 t^2} dt \\
&= \frac{\pi}{\alpha} - \frac{2(1+\alpha)}{\alpha(1-\alpha)} \left[\frac{\alpha-1}{\alpha+1} \tan^{-1} \left(\frac{\alpha+1}{\alpha-1} t \right) \right]_0^\infty = \frac{\pi}{\alpha} + \frac{\pi}{\alpha} = \frac{2\pi}{\alpha}
\end{aligned}$$

$$I(\alpha) = \int \frac{2\pi}{\alpha} d\alpha = 2\pi \ln |\alpha| + C, \quad I(1) = \int_0^\pi \ln(2-2\cos x) dx = 0 = C \quad (*)$$

$$I(\alpha) = \int_0^\pi \ln(1-2\alpha \cos x + \alpha^2) dx = 2\pi \ln |\alpha|$$

$$\therefore \int_0^\pi \ln(1-2e \cos x + e^2) dx = I(e) = 2\pi \quad \blacksquare$$

$$* \quad J = \int_0^\pi \ln(2-2\cos x) dx = \pi \ln 2 + \int_0^\pi \ln(1-\cos x) dx$$

$$= \pi \ln 2 + \int_0^\pi \ln(1+\cos x) dx \quad (x \mapsto \pi-x)$$

$$= \pi \ln 2 + \frac{1}{2} \int_0^\pi \ln(1-\cos^2 x) dx = \pi \ln 2 + \frac{1}{2} \int_0^\pi \ln(\sin^2 x) dx$$

$$= \pi \ln 2 + \int_0^\pi \ln(\sin x) dx = \pi \ln 2 + 2K$$

$$K = \frac{1}{2} \int_0^\pi \ln(\sin x) dx = \int_0^{\frac{\pi}{2}} \ln(\sin x) dx = \int_0^{\frac{\pi}{2}} \ln(\cos x) dx \quad (x \mapsto \frac{\pi}{2}-x)$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln(\sin x \cos x) dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{2} \sin 2x\right) dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln(\sin 2x) dx - \frac{\pi}{4} \ln 2$$

$$= \frac{1}{4} \int_0^{\pi} \ln(\sin t) dt - \frac{\pi}{4} \ln 2 \quad (t = 2x, \quad dt = 2dx)$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln(\sin t) dt - \frac{\pi}{4} \ln 2 = \frac{1}{2} I - \frac{\pi}{4} \ln 2, \quad \frac{1}{2} I = -\frac{\pi}{4} \ln 2$$

$$\therefore K = \int_0^{\frac{\pi}{2}} \ln(\sin x) dx = -\frac{\pi}{2} \ln 2$$

$$\therefore J = \pi \ln 2 + 2K = 0 \quad \blacksquare$$

55. $x = \tan t, \quad dx = \sec^2 t dt$

$$I = \int_0^{\pi} \ln(1 + \sin^2 t) dt = 2 \int_0^{\frac{\pi}{2}} \ln(1 + \sin^2 t) dt = 2 \int_0^{\infty} \frac{\ln(2 + x^2) - \ln(1 + x^2)}{1 + x^2} dx$$

$$I(\alpha) := \int_0^{\infty} \frac{\ln(\alpha + x^2) - \ln(1 + x^2)}{1 + x^2} dx,$$

$$\frac{d}{d\alpha} I(\alpha) = \int_0^{\infty} \frac{\partial}{\partial \alpha} \frac{\ln(\alpha + x^2) - \ln(1 + x^2)}{1 + x^2} dx = \int_0^{\infty} \frac{1}{(1 + x^2)(\alpha + x^2)} dx$$

$$= \frac{1}{\alpha - 1} \left(\int_0^{\infty} \frac{1}{x^2 + 1} dx - \int_0^{\infty} \frac{1}{x^2 + \alpha} dx \right) = \frac{1}{\alpha - 1} \left([\tan^{-1} x]_0^{\infty} - \left[\frac{1}{\sqrt{\alpha}} \tan^{-1} \left(\frac{x}{\sqrt{\alpha}} \right) \right]_0^{\infty} \right)$$

$$= \frac{\pi}{2\sqrt{\alpha}(1 + \sqrt{\alpha})}$$

한편 $I(1) = \int_0^{\infty} 0 dx = 0$ 이므로

$$I = 2I(2) = 2 \int_1^2 \frac{d}{d\alpha} I(\alpha) d\alpha = \pi \int_1^2 \frac{1}{\sqrt{\alpha}(1 + \sqrt{\alpha})} d\alpha$$

$$= 2\pi \int_1^{\sqrt{2}} \frac{1}{1 + u} du \quad (\alpha = u^2, \quad d\alpha = 2u du)$$

$$= 2\pi [\ln|1+u|]_1^{\sqrt{2}} = 2\pi \ln\left(\frac{1+\sqrt{2}}{2}\right) \blacksquare$$

56. $x = u + 2, dx = du$

$$\begin{aligned} I &= \int_0^4 \frac{\ln x}{\sqrt{4x-x^2}} dx = \int_{-2}^2 \frac{\ln(u+2)}{\sqrt{4-u^2}} du = \int_0^2 \frac{\ln(u+2)}{\sqrt{4-u^2}} du + \int_{-2}^0 \frac{\ln(u+2)}{\sqrt{4-u^2}} du \\ &= \int_0^2 \frac{\ln(u+2)}{\sqrt{4-u^2}} du + \int_2^0 \frac{\ln(-u+2)}{\sqrt{4-u^2}} (-du) \quad (u \mapsto -u) \\ &= \int_0^2 \frac{\ln(4-u^2)}{\sqrt{4-u^2}} du \end{aligned}$$

한편 $x = 4 - u^2, dx = -2udu$ 의 치환을 하면

$$I = \int_0^4 \frac{\ln x}{\sqrt{4x-x^2}} dx = \int_2^0 \frac{\ln(4-u^2)}{u\sqrt{4-u^2}} \cdot (-2udu) = 2 \int_0^2 \frac{\ln(4-u^2)}{\sqrt{4-u^2}} du$$

$$I = 2I, \therefore I = \int_0^4 \frac{\ln x}{\sqrt{4x-x^2}} dx = 0 \blacksquare$$

57. $x = \tan t, dx = \sec^2 t dt, \frac{dx}{1+x^2} = dt$

$$\begin{aligned} I &= \int_0^\infty \frac{1}{(1+x^2)(1+x^\pi)} dx = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^\pi t} dt \\ &= \int_{\frac{\pi}{2}}^0 \frac{1}{1+\tan^\pi\left(\frac{\pi}{2}-t\right)} (-dt) = \int_0^{\frac{\pi}{2}} \frac{1}{1+\cot^\pi t} dt = \int_0^{\frac{\pi}{2}} \frac{\tan^\pi t}{1+\tan^\pi t} dt \quad (t \mapsto \frac{\pi}{2}-t) \end{aligned}$$

$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{1}{1+\tan^\pi t} + \frac{\tan^\pi t}{1+\tan^\pi t} \right) dt = \frac{\pi}{2}$$

$$\therefore I = \int_0^\infty \frac{1}{(1+x^2)(1+x^\pi)} dx = \frac{\pi}{4} \blacksquare$$

$$58. \text{ sol 1) } I(v) := \int_0^\infty \frac{1}{(v+x)(\pi^2 + (\ln x)^2)} dx, \quad J(v) := \int_{-\infty}^\infty \frac{1}{(v+e^t)(\pi^2 + t^2)} dt$$

$$x = e^t, \quad dx = e^t dt$$

$$I(v) = \int_{-\infty}^\infty \frac{e^t}{(v+e^t)(\pi^2 + t^2)} dt = \int_{-\infty}^\infty \frac{(v+e^t)}{(v+e^t)(\pi^2 + t^2)} dt - v \int_{-\infty}^\infty \frac{1}{(v+e^t)(\pi^2 + t^2)} dt$$

$$= \left[\frac{1}{\pi} \tan^{-1} \left(\frac{x}{\pi} \right) \right]_{-\infty}^\infty - v J(v) = 1 - v J(v)$$

$$t = -z, \quad dt = -dz$$

$$I(v) = \int_{\infty}^{-\infty} \frac{e^{-z}}{(v+e^{-z})(\pi^2 + z^2)} (-dz) = \int_{-\infty}^\infty \frac{1}{(1+ve^z)(\pi^2 + z^2)} dz = \frac{1}{v} J\left(\frac{1}{v}\right)$$

$$1 - v J(v) = I(v) = \frac{1}{v} J\left(\frac{1}{v}\right), \quad v J(v) + \frac{1}{v} J\left(\frac{1}{v}\right) = 1$$

$$v = 1, \quad \therefore J(1) = I(1) = \int_0^\infty \frac{1}{(1+x)(\pi^2 + (\ln x)^2)} dx = \frac{1}{2} \blacksquare$$

$$\text{sol 2) } I = \int_0^\infty \frac{1}{(1+x)(\pi^2 + (\ln x)^2)} dx$$

$$= \int_0^1 \frac{1}{(1+x)(\pi^2 + (\ln x)^2)} dx + \int_1^\infty \frac{1}{(1+x)(\pi^2 + (\ln x)^2)} dx$$

$$= \int_0^1 \frac{1}{(1+x)(\pi^2 + (\ln x)^2)} dx + \int_1^0 \frac{1}{(1+1/x)(\pi^2 + (\ln x)^2)} \cdot \left(-\frac{1}{x^2}\right) dx \quad (x \mapsto \frac{1}{x})$$

$$= \int_0^1 \frac{1}{(1+x)(\pi^2 + (\ln x)^2)} dx + \int_0^1 \frac{1}{x(1+x)(\pi^2 + (\ln x)^2)} dx$$

$$= \int_0^1 \left(\frac{1}{(1+x)(\pi^2 + (\ln x)^2)} \cdot \frac{x+1}{x} \right) dx = \int_0^1 \frac{1}{x(\pi^2 + (\ln x)^2)} dx$$

$$= \int_{-\infty}^0 \frac{1}{\pi^2 + u^2} du \quad (u = \ln x, \quad du = \frac{dx}{x})$$

$$= \left[\frac{1}{\pi} \tan^{-1} \left(\frac{u}{\pi} \right) \right]_{-\infty}^0 = \frac{1}{2} \blacksquare$$

$$\begin{aligned} 59. \quad I &= \int_0^{\infty} \frac{x-1}{\sqrt{2^x-1} \ln(2^x-1)} dx \\ &= \frac{1}{(\ln 2)^2} \int_0^{\infty} \frac{\ln(t^2+1) - \ln 2}{(t^2+1)\ln t} dt \quad (t = \sqrt{2^x-1}, \quad dt = \frac{2^{x-1} \ln 2}{\sqrt{2^x-1}} dx) \\ &= \frac{1}{(\ln 2)^2} \int_{\infty}^0 \frac{\ln(u^{-2}+1) - \ln 2}{\left(\frac{1}{u^2}+1\right)(-\ln u)} \left(-\frac{1}{u^2}\right) du = -\frac{1}{(\ln 2)^2} \int_0^{\infty} \frac{\ln(u^2+1) - \ln u^2 - \ln 2}{(u^2+1)\ln u} du \end{aligned}$$

$$(u = \frac{1}{t}, \quad dt = -\frac{1}{u^2} du)$$

$$= -\frac{1}{(\ln 2)^2} \int_0^{\infty} \frac{\ln(u^2+1) - \ln 2}{(u^2+1)\ln u} du + \frac{2}{(\ln 2)^2} \int_0^{\infty} \frac{1}{u^2+1} du$$

$$= -I + \frac{2}{(\ln 2)^2} [\tan^{-1} u]_0^{\infty} = -I + \frac{\pi}{(\ln 2)^2}$$

$$\therefore I = \int_0^{\infty} \frac{x-1}{\sqrt{2^x-1} \ln(2^x-1)} dx = \frac{\pi}{2(\ln 2)^2} \blacksquare$$

$$60. \quad t = \tan x, \quad dt = \sec^2 x dx$$

$$\begin{aligned} \int_0^{3\pi} \frac{1}{\sin^4 x + \cos^4 x} dx &= 6 \int_0^{\frac{\pi}{2}} \frac{\sec^4 x}{1 + \tan^4 x} dx = 6 \int_0^{\infty} \frac{1+t^2}{1+t^4} dt \\ &= 6 \int_0^{\infty} \left(\frac{1}{2(t^2 + \sqrt{2}t + 1)} - \frac{1}{2(-t^2 + \sqrt{2}t - 1)} \right) dt \\ &= 3 \int_0^{\infty} \frac{1}{t^2 + \sqrt{2}t + 1} dt + 3 \int_0^{\infty} \frac{1}{t^2 - \sqrt{2}t + 1} dt \\ &= 3 \int_0^{\infty} \frac{1}{\left(t + \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} dt + 3 \int_0^{\infty} \frac{1}{\left(t - \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} dt \\ &= 3 [\sqrt{2} \tan^{-1}(\sqrt{2}t + 1)]_0^{\infty} + 3 [\sqrt{2} \tan^{-1}(\sqrt{2}t - 1)]_0^{\infty} = 3\sqrt{2}\pi \blacksquare \end{aligned}$$

$$\begin{aligned}
61. \int_0^\pi \cos^4 x dx &= \frac{1}{4} \int_0^\pi (1 - \cos 2x)^2 dx = \frac{1}{4} \int_0^\pi \left(1 - 2\cos 2x + \frac{1 + \cos 4x}{2}\right) dx \\
&= \int_0^\pi \left(\frac{3}{8} - \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x\right) dx = \left[\frac{3}{8}x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x\right]_0^\pi = \frac{3}{8}\pi \blacksquare
\end{aligned}$$

$$\begin{aligned}
62. \int_0^{\frac{\pi}{2}} \sin^6 x \cos^3 x dx &= \int_0^{\frac{\pi}{2}} \sin^6 x (1 - \sin^2 x) \cos x dx \\
&= \int_0^{\frac{\pi}{2}} \sin^6 x \cos x dx - \int_0^{\frac{\pi}{2}} \sin^8 x \cos x dx = \left[\frac{1}{7} \sin^7 x - \frac{1}{9} \sin^9 x\right]_0^{\frac{\pi}{2}} = \frac{2}{63} \blacksquare
\end{aligned}$$

$$\begin{aligned}
63. \int_0^{\frac{\pi}{4}} \sin^3 x \sec^2 x dx &= \int_0^{\frac{\pi}{4}} \frac{\sin x (1 - \cos^2 x)}{\cos^2 x} dx = \int_0^{\frac{\pi}{4}} (\tan x \sec x - \sin x) dx \\
&= [\sec x + \cos x]_0^{\frac{\pi}{4}} = \frac{3\sqrt{2}}{2} - 2 \blacksquare
\end{aligned}$$

$$64. x = 2 \tan t, \quad dx = 2 \sec^2 t dt$$

$$\begin{aligned}
\int_1^3 \frac{1}{x^2 \sqrt{x^2 + 4}} dx &= \int_\alpha^\beta \frac{\operatorname{sect} t}{4 \tan^2 t} dt = \frac{1}{4} \int_\alpha^\beta \operatorname{csct} \cot t dt = \left[-\frac{1}{4} \operatorname{csct}\right]_\alpha^\beta = \left[-\frac{\sqrt{x^2 + 4}}{4x}\right]_1^3 \\
&= \frac{3\sqrt{5} - \sqrt{13}}{12} \blacksquare \quad \left(\alpha = \tan^{-1}\left(\frac{1}{2}\right), \beta = \tan^{-1}\left(\frac{3}{2}\right)\right)
\end{aligned}$$

$$65. x + 1 = \tan t, \quad dx = \sec^2 t dt$$

$$\begin{aligned}
\int_{\sqrt{3}-1}^{2\sqrt{2}-1} \frac{1}{(x+1)\sqrt{x^2+2x+2}} dx &= \int_\alpha^\beta \frac{\sec^2 t}{\tan t \operatorname{sect} t} dt = \int_\alpha^\beta \operatorname{csct} dt = [\ln |\operatorname{csct} - \cot t|]_\alpha^\beta \\
&= \left[\ln \left| \frac{\sqrt{x^2+2x+2}-1}{x+1} \right| \right]_{\sqrt{3}-1}^{2\sqrt{2}-1} = \ln \sqrt{\frac{3}{2}} \blacksquare \quad \left(\alpha = \frac{\pi}{3}, \beta = \tan^{-1}(2\sqrt{2})\right)
\end{aligned}$$

$$66. \int_0^{\tan^{-1}(\sqrt{6})} \frac{1}{1 + \cos^2 x} dx = \int_0^{\tan^{-1}(\sqrt{6})} \sec^2 x \cdot \frac{1}{\tan^2 x + 2} dx$$

$$= \left[\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\tan x}{\sqrt{2}} \right) \right]_0^{\tan^{-1}(\sqrt{6})} = \frac{\sqrt{2}}{6} \pi \blacksquare$$

$$67. \text{ sol 1) } \int_0^{\frac{\pi}{2}} \frac{1}{1 + \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{1 - \cos x}{\sin^2 x} dx = \int_0^{\frac{\pi}{2}} (\csc^2 x - \csc x \cot x) dx$$

$$= [\csc x - \cot x]_0^{\frac{\pi}{2}} = 1 - \lim_{x \rightarrow 0} (\csc x - \cot x) = 1 \blacksquare$$

$$\text{sol 2) } \int_0^{\frac{\pi}{2}} \frac{1}{1 + \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{1}{2 \cos^2 \frac{x}{2}} dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \sec^2 \frac{x}{2} dx = \left[\tan \frac{x}{2} \right]_0^{\frac{\pi}{2}} = 1 \blacksquare$$

$$68. t = e^x + 1, dt = e^x dx$$

$$\int_{\ln(e-1)}^{\ln(e^3-1)} \frac{e^x \ln(e^x + 1)}{e^x + 1} dx = \int_e^{e^3} \frac{\ln t}{t} dt = \left[\frac{1}{2} (\ln t)^2 \right]_e^{e^3} = 4 \blacksquare$$

$$69. \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{\sin x} dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \csc x dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x} dx$$

$$= [\ln |\csc x - \cot x|]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \ln \sqrt{3} \blacksquare$$

$$70. \int_0^{\frac{\pi}{4}} \frac{1}{1 - 3 \cos^2 x} dx = \int_0^{\frac{\pi}{4}} \sec^2 x \cdot \frac{1}{\tan^2 x - 2} dx = \left[\frac{1}{2\sqrt{2}} \ln \left| \frac{\tan x - \sqrt{2}}{\tan x + \sqrt{2}} \right| \right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2\sqrt{2}} \ln(3 - 2\sqrt{2}) = \frac{1}{\sqrt{2}} \ln(\sqrt{2} - 1) \blacksquare$$

$$71. \int_{-1}^0 \frac{x^3 - x - 2}{x^3 - x^2 + x - 1} dx = \int_{-1}^0 \left(1 - \frac{1}{x-1} + \frac{2x}{x^2+1} \right) dx = [x - \ln|x-1| + \ln(x^2+1)]_{-1}^0$$

$$= 1 \blacksquare$$

$$72. \int_3^4 \frac{x+4}{x^3 + 3x^2 - 10x} dx = \int_3^4 \frac{x+4}{x(x+5)(x-2)} dx = \int_3^4 \left(-\frac{2}{5x} - \frac{1}{35(x+5)} + \frac{3}{7(x-2)} \right) dx$$

$$= \left[-\frac{2}{5} \ln|x| - \frac{1}{35} \ln|x+5| + \frac{3}{7} \ln|x-2| \right]_3^4 = \frac{12}{35} \ln 3 - \frac{10}{35} \ln 2 = \frac{2}{35} \ln \left(\frac{729}{32} \right) \blacksquare$$

$$\begin{aligned}
73. \quad \int_5^6 \frac{7x^3 - 13x^2 - 24x + 24}{x^4 - 3x^3 - 10x^2 + 24x} dx &= \int_5^6 \frac{7x^3 - 13x^2 - 24x + 24}{x(x-2)(x+3)(x-4)} dx \\
&= \int_5^6 \left(\frac{1}{x-2} + \frac{1}{x} + \frac{2}{x+3} + \frac{3}{x-4} \right) dx = [\ln|x-2| + \ln|x| + 2\ln|x+3| + 3\ln|x-4|]_5^6 \\
&= 4\ln 3 - \ln 5 = \ln\left(\frac{81}{5}\right) \blacksquare
\end{aligned}$$

$$74. \quad t = \frac{1}{x}, \quad dt = -\frac{1}{x^2} dx$$

$$\begin{aligned}
\int_1^2 \frac{1}{x^2 \sqrt{2x-x^2}} dx &= \int_1^2 \frac{1}{x^3 \sqrt{\frac{2}{x}-1}} dx = -\int_1^{\frac{1}{2}} \frac{t}{\sqrt{2t-1}} dt = \frac{1}{2} \int_{\frac{1}{2}}^1 \frac{2t-1+1}{\sqrt{2t-1}} dt \\
&= \frac{1}{2} \int_{\frac{1}{2}}^1 \left(\sqrt{2t-1} + \frac{1}{\sqrt{2t-1}} \right) dt = \frac{1}{2} \left[\frac{1}{6} (2t-1)^{\frac{3}{2}} + \frac{1}{2} \sqrt{2t-1} \right]_{\frac{1}{2}}^1 = \frac{2}{3} \blacksquare
\end{aligned}$$

$$75. \quad t = x^2, \quad dt = 2x dx$$

$$\begin{aligned}
\int_1^e \frac{x^4 + 81}{x(x^2 + 9)^2} dx &= \frac{1}{2} \int_1^{e^2} \frac{t^2 + 81}{t(t+9)^2} dt = \frac{1}{2} \int_1^{e^2} \left(\frac{1}{t} - \frac{18}{(t+9)^2} \right) dt \\
&= \left[\frac{1}{2} \ln|t| + \frac{9}{t+9} \right]_1^{e^2} = \frac{e^2 + 99}{10(e^2 + 9)} \blacksquare
\end{aligned}$$

$$76. \quad t = \sqrt{x}, \quad dt = \frac{1}{2\sqrt{x}} dx, \quad t = 2\sin\theta, \quad dt = 2\cos\theta d\theta, \quad \sin\theta = \frac{\sqrt{x}}{2}$$

$$\begin{aligned}
\int_0^3 \sqrt{\frac{4-x}{x}} dx &= \int_0^{\sqrt{3}} \frac{\sqrt{4-t^2}}{t} \cdot 2t dt = 2 \int_0^{\sqrt{3}} \sqrt{4-t^2} dt = 2 \int_0^{\frac{\pi}{3}} 4\cos^2\theta d\theta \\
&= 4 \int_0^{\frac{\pi}{3}} (1 + \cos 2\theta) d\theta = [4\theta + 4\sin\theta\cos\theta]_0^{\frac{\pi}{3}} = \sqrt{3} + \frac{4}{3}\pi \blacksquare
\end{aligned}$$

$$77. \quad t = x^{\frac{3}{2}}, \quad dt = \frac{3}{2} \sqrt{x} dx$$

$$\int_0^{\frac{1}{\sqrt[3]{2}}} \sqrt{\frac{x}{1-x^3}} dx = \frac{2}{3} \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-t^2}} dt = \left[\frac{2}{3} \sin^{-1} t \right]_0^{\frac{1}{\sqrt{2}}} = \frac{\pi}{6} \blacksquare$$

78. $t = \sqrt{x}$, $dt = \frac{1}{2\sqrt{x}} dx$, $t = \sin u$, $dt = \cos u du$

$$\begin{aligned} \int_0^{\frac{3}{4}} \sqrt{x} \sqrt{1-x} dx &= \int_0^{\frac{\sqrt{3}}{2}} t \sqrt{1-t^2} \cdot 2t dt = 2 \int_0^{\frac{\sqrt{3}}{2}} t^2 \sqrt{1-t^2} dt \\ &= 2 \int_0^{\frac{\pi}{3}} \sin^2 u \cos u \cdot \cos u du = \frac{1}{2} \int_0^{\frac{\pi}{3}} \sin^2 2u du = \frac{1}{4} \int_0^{\frac{\pi}{3}} (1 - \cos 4u) du \\ &= \left[\frac{1}{4} u - \frac{1}{16} \sin 4u \right]_0^{\frac{\pi}{3}} = \frac{\sqrt{3}}{32} + \frac{\pi}{12} \blacksquare \end{aligned}$$

79. $t = \tan x$, $dt = \sec^2 x dx$

$$\begin{aligned} \int_0^{\tan^{-1}(\sqrt[4]{3})} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx &= \int_0^{\tan^{-1}(\sqrt[4]{3})} \frac{\tan x \sec^2 x}{1 + \tan^4 x} dx = \int_0^{\sqrt[4]{3}} \frac{t}{1+t^4} dt \\ &= \left[\frac{1}{2} \tan^{-1}(t^2) \right]_0^{\sqrt[4]{3}} = \frac{\pi}{6} \blacksquare \end{aligned}$$

80. $\int_0^{\frac{\pi}{4}} \tan^4 x \sec^4 x dx = \int_0^{\frac{\pi}{4}} \tan^4 x (\tan^2 x + 1) \sec^2 x dx$

$$= \int_0^{\frac{\pi}{4}} \tan^6 x \sec^2 x dx + \int_0^{\frac{\pi}{4}} \tan^4 x \sec^2 x dx = \left[\frac{1}{7} \tan^7 x + \frac{1}{5} \tan^5 x \right]_0^{\frac{\pi}{4}} = \frac{12}{35} \blacksquare$$

81. $t = \sqrt{x}$, $dt = \frac{1}{2\sqrt{x}} dx$

$$\begin{aligned} \int_2^4 \frac{x\sqrt{x}-1}{x-\sqrt{x}} dx &= \int_{\sqrt{2}}^2 \frac{t^3-1}{t^2-t} \cdot 2t dt = 2 \int_{\sqrt{2}}^2 (t^2+t+1) dt = \left[\frac{2}{3} t^3 + t^2 + 2t \right]_{\sqrt{2}}^2 \\ &= \frac{34-10\sqrt{2}}{3} \blacksquare \end{aligned}$$

$$82. \quad t = e^x, \quad dt = e^x dx, \quad u = \sqrt{t+1}, \quad du = \frac{1}{2\sqrt{t+1}} dt$$

$$\int_0^{\ln 3} \frac{1}{\sqrt{1+e^x}} dx = \int_1^3 \frac{1}{t\sqrt{t+1}} dt = \int_{\sqrt{2}}^2 \frac{2}{u^2-1} du = \left[\ln \left| \frac{u-1}{u+1} \right| \right]_{\sqrt{2}}^2$$

$$= -\ln(9-6\sqrt{2}) \blacksquare$$

$$83. \quad \int_1^2 \frac{x^2+2x-1}{2x^3+3x^2-2x} dx = \int_1^2 \frac{x^2+2x-1}{x(x+2)(2x-1)} dx = \int_1^2 \left(-\frac{1}{10(x+2)} + \frac{1}{5(2x-1)} + \frac{1}{2x} \right) dx$$

$$= \left[-\frac{1}{10} \ln|x+2| + \frac{1}{10} \ln|2x-1| + \frac{1}{2} \ln|x| \right]_1^2 = \frac{3}{10} \ln 2 + \frac{1}{5} \ln 3 = \frac{1}{10} \ln 72 \blacksquare$$

$$84. \quad I = \int_0^{\frac{\pi}{3}} 13e^{2x} \cos 3x dx = \left[\frac{13}{2} e^{2x} \cos 3x \right]_0^{\frac{\pi}{3}} + \frac{3}{2} \int_0^{\frac{\pi}{3}} 13e^{2x} \sin 3x dx$$

$$= -\frac{13}{2} (e^{2\pi/3} + 1) + \frac{39}{2} \left\{ \left[\frac{1}{2} e^{2x} \sin 3x \right]_0^{\frac{\pi}{3}} - \frac{1}{2} \int_0^{\frac{\pi}{3}} 3e^{2x} \cos 3x dx \right\}$$

$$= -\frac{13}{2} (e^{2\pi/3} + 1) - \frac{9}{4} I$$

$$\frac{13}{4} I = -\frac{13}{2} (e^{2\pi/3} + 1)$$

$$\therefore I = \int_0^{\frac{\pi}{3}} 13e^{2x} \cos 3x dx = -2(e^{2\pi/3} + 1) \blacksquare$$

$$85. \quad \int_0^{\pi} e^{-x} \sin^2 2x dx = \frac{1}{2} \int_0^{\pi} e^{-x} (1 - \cos 4x) dx = \frac{1}{2} [-e^{-x}]_0^{\pi} - \frac{1}{2} \int_0^{\pi} e^{-x} \cos 4x dx$$

$$= \frac{1-e^{-\pi}}{2} - \frac{1}{2} \left(\left[\frac{1}{4} e^{-x} \sin 4x \right]_0^{\pi} + \frac{1}{4} \int_0^{\pi} e^{-x} \sin 4x dx \right)$$

$$= \frac{1-e^{-\pi}}{2} - \frac{1}{8} \left(\left[-\frac{1}{4} e^{-x} \cos 4x \right]_0^{\pi} - \frac{1}{4} \int_0^{\pi} e^{-x} \cos 4x dx \right)$$

$$= \frac{15(1-e^{-\pi})}{32} + \frac{1}{32} \int_0^{\pi} e^{-x} \cos 4x dx$$

$$\frac{17}{32} \int_0^\pi e^{-x} \cos 4x dx = \frac{1-e^{-\pi}}{32}, \quad \int_0^\pi e^{-x} \cos 4x dx = \frac{1-e^{-\pi}}{17}$$

$$\therefore \int_0^\pi e^{-x} \sin^2 2x dx = \frac{1-e^{-\pi}}{2} - \frac{1}{2} \int_0^\pi e^{-x} \cos 4x dx = \frac{8}{17}(1-e^{-\pi}) \blacksquare$$

86. sol 1) $t^2 = \tan x$, $2t dt = \sec^2 x dx$, $dx = \frac{2t}{t^4+1} dt$

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \sqrt{\tan x} dx &= \int_0^1 \frac{2t^2}{t^4+1} dt = \int_0^1 \frac{2}{t^2+\frac{1}{t^2}} dt = \int_0^1 \frac{\left(1+\frac{1}{t^2}\right) + \left(1-\frac{1}{t^2}\right)}{t^2+\frac{1}{t^2}} dt \\ &= \int_0^1 \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}} dt + \int_0^1 \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}} dt = \int_0^1 \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2 + (\sqrt{2})^2} dt + \int_0^1 \frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2 - (\sqrt{2})^2} dt \\ &= \left[\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{t-1/t}{\sqrt{2}} \right) + \frac{1}{2\sqrt{2}} \ln \left| \frac{t+1/t-\sqrt{2}}{t+1/t+\sqrt{2}} \right| \right]_0^1 = \frac{\pi}{2\sqrt{2}} + \frac{1}{\sqrt{2}} \ln(\sqrt{2}-1) \blacksquare \end{aligned}$$

sol 2) $I := \int_0^{\frac{\pi}{4}} \sqrt{\tan x} dx$, $J := \int_0^{\frac{\pi}{4}} \sqrt{\cot x} dx$

$$\begin{aligned} I+J &= \int_0^{\frac{\pi}{4}} (\sqrt{\tan x} + \sqrt{\cot x}) dx = \sqrt{2} \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx \\ &= \sqrt{2} \int_0^{\frac{\pi}{4}} \frac{(\sin x - \cos x)'}{\sqrt{1 - (\sin x - \cos x)^2}} dx = \sqrt{2} [\sin^{-1}(\sin x - \cos x)]_0^{\frac{\pi}{4}} = \frac{\pi}{\sqrt{2}} \quad \dots [1] \end{aligned}$$

$$\begin{aligned} I-J &= \int_0^{\frac{\pi}{4}} (\sqrt{\tan x} - \sqrt{\cot x}) dx = \sqrt{2} \int_0^{\frac{\pi}{4}} \frac{\sin x - \cos x}{\sqrt{\sin 2x}} dx \\ &= -\sqrt{2} \int_0^{\frac{\pi}{4}} \frac{(\sin x + \cos x)'}{\sqrt{(\sin x + \cos x)^2 - 1}} dx \\ &= \left[-\sqrt{2} \ln \left| (\sin x + \cos x) + \sqrt{(\sin x + \cos x)^2 - 1} \right| \right]_0^{\frac{\pi}{4}} \end{aligned}$$

$$= -\sqrt{2} \ln(\sqrt{2}+1) = \sqrt{2} \ln(\sqrt{2}-1) \dots [2]$$

$$\frac{[1]+[2]}{2} : \therefore I = \int_0^{\frac{\pi}{4}} \sqrt{\tan x} dx = \frac{\pi}{2\sqrt{2}} + \frac{1}{\sqrt{2}} \ln(\sqrt{2}-1) \blacksquare$$

87. $t = \tan x$, $dt = \sec^2 x dx$, $u = \frac{t}{1-t}$, $du = \frac{1}{(1-t)^2} dt$, $u = y^2$, $du = 2y dy$

$$\begin{aligned} I &= \int_0^{\frac{\pi}{4}} \sqrt{\tan x} \sqrt{1-\tan x} dx = \int_0^1 \frac{\sqrt{t(1-t)}}{1+t^2} dt = \int_0^\infty \frac{\sqrt{u}}{(1+u)(1+2u+2u^2)} du \\ &= \int_0^\infty \frac{2y^2}{(1+y^2)(1+2y^2+2y^4)} dy = 2 \int_0^\infty \left(\frac{2y^2}{1+2y^2+2y^4} + \frac{1}{1+2y^2+2y^4} - \frac{1}{1+y^2} \right) dy \\ &= \int_{-\infty}^\infty \left(\frac{2y^2}{1+2y^2+2y^4} + \frac{1}{1+2y^2+2y^4} - \frac{1}{1+y^2} \right) dy = I_1 + I_2 - [\tan^{-1} y]_{-\infty}^\infty = I_1 + I_2 - \pi \end{aligned}$$

$$I_1 = \int_{-\infty}^\infty \frac{2y^2}{1+2y^2+2y^4} dy = \int_{-\infty}^\infty \frac{1}{y^2 + \frac{1}{2y^2} + 1} dy = \int_{-\infty}^\infty \frac{1}{\left(y - \frac{1}{\sqrt{2}y}\right)^2 + 1 + \sqrt{2}} dy$$

$$= \int_{-\infty}^\infty \frac{1}{y^2 + 1 + \sqrt{2}} dy \quad (\because \text{Glasser's Master Theorem})$$

$$= \left[\frac{1}{\sqrt{1+\sqrt{2}}} \tan^{-1} \left(\frac{y}{\sqrt{1+\sqrt{2}}} \right) \right]_{-\infty}^\infty = \frac{\pi}{\sqrt{1+\sqrt{2}}}$$

$$I_2 = \frac{1}{2} \int_{-\infty}^\infty \frac{1}{y^4 + y^2 + \frac{1}{2}} dy = \int_0^\infty \frac{1}{y^2 + \frac{1}{2y^2} + 1} \cdot \frac{dy}{y^2} = \int_0^\infty \frac{1}{\frac{y^2}{2} + \frac{1}{y^2} + 1} dy \quad (y \mapsto \frac{1}{y})$$

$$= \int_{-\infty}^\infty \frac{1}{y^2 + \frac{2}{y^2} + 2} dy = \int_{-\infty}^\infty \frac{1}{\left(y - \frac{\sqrt{2}}{y}\right)^2 + 2 + 2\sqrt{2}}$$

$$= \int_{-\infty}^\infty \frac{1}{y^2 + 2 + 2\sqrt{2}} dy \quad (\because \text{Glasser's Master Theorem})$$

$$= \left[\frac{1}{\sqrt{2+2\sqrt{2}}} \tan^{-1} \left(\frac{y}{\sqrt{2+2\sqrt{2}}} \right) \right]_{-\infty}^{\infty} = \frac{\pi}{\sqrt{2+2\sqrt{2}}}$$

$$\therefore I = \int_0^{\frac{\pi}{4}} \sqrt{\tan x} \sqrt{1-\tan x} dx = I_1 + I_2 - \pi = \left(\sqrt{\frac{1+\sqrt{2}}{2}} - 1 \right) \pi \blacksquare$$

$$88. I_{m,n} := \int_1^{\infty} \frac{(\ln x)^m}{x^n} dx = \int_1^{\infty} \frac{(\ln x)^m}{x} \cdot \frac{1}{x^{n-1}} dx$$

$$= \left[\frac{(\ln x)^{m+1}}{(m+1)x^{n-1}} \right]_1^{\infty} - \frac{1-n}{m+1} \int_1^{\infty} \frac{(\ln x)^{m+1}}{x^n} dx = \frac{n-1}{m+1} \int_1^{\infty} \frac{(\ln x)^{m+1}}{x^n} dx$$

$$= \frac{n-1}{m+1} I_{m+1,n}, \quad I_{m,n} = \frac{m}{n-1} I_{m-1,n} \text{ 을 얻는다. 한편}$$

$$I_{0,n} = \int_1^{\infty} \frac{1}{x^n} dx = \frac{1}{n-1} \text{ 이므로}$$

$$I_{m,n} = \frac{m!}{(n-1)^{m+1}}, \quad m, n \in \mathbb{N}_0.$$

$$\therefore \int_1^{\infty} \frac{(\ln x)^{627}}{x^{2022}} dx = I_{627,2022} = \frac{627!}{2021^{628}} \blacksquare$$

$$89. x = \cot t, \quad \sin(\cot^{-1} x) = \sin t = \frac{x}{\sqrt{1+x^2}}, \quad x = \tan u, \quad \cos(\tan^{-1} x) = \cos u = \frac{1}{\sqrt{1+x^2}}$$

$$\int_0^{\sqrt{6}} \cos^2(\tan^{-1}(\sin(\cot^{-1} x))) dx = \int_0^{\sqrt{6}} \cos^2\left(\tan^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right) dx$$

$$= \int_0^{\sqrt{6}} \frac{1+x^2}{2+x^2} dx = \int_0^{\sqrt{6}} \left(1 - \frac{1}{2+x^2}\right) dx = \left[x - \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) \right]_0^{\sqrt{6}} = \sqrt{6} - \frac{\pi}{3\sqrt{2}} \blacksquare$$

$$90. t = \tan x, \quad dt = \sec^2 x dx, \quad u = t + \sqrt{1+t^2}, \quad t = \frac{1}{2}\left(u - \frac{1}{u}\right), \quad dt = \frac{1}{2}\left(1 + \frac{1}{u^2}\right) du$$

$$\int_0^{\frac{\pi}{6}} \frac{\sec^2 x}{(\sec x + \tan x)^{5/2}} dx = \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{(t + \sqrt{1+t^2})^{5/2}} dt = \frac{1}{2} \int_1^{\sqrt{3}} (u^{-5/2} + u^{-9/2}) du$$

$$= \left[-\frac{1}{3}u^{-3/2} - \frac{1}{7}u^{-7/2} \right]_1^{\sqrt{3}} = -\frac{8}{63}\sqrt[4]{3} + \frac{10}{21} \blacksquare$$

91. 제 1종 오일러 치환 : $\sqrt{x^2+4x-4} = x+t$, $x = \frac{t^2+4}{4-2t}$, $dx = \frac{-2t^2+8t+8}{(4-2t)^2} dt$

$$\begin{aligned} \int_1^2 \frac{1}{x\sqrt{x^2+4x-4}} dx &= \int_0^{2\sqrt{2}-2} \frac{4-2t}{t^2+4} \cdot \frac{4-2t}{-t^2+4t+4} \cdot \frac{-2t^2+8t+8}{(4-2t)^2} dt \\ &= 2 \int_0^{2\sqrt{2}-2} \frac{1}{t^2+4} dt = \left[\tan^{-1}\left(\frac{t}{2}\right) \right]_0^{2\sqrt{2}-2} = \tan^{-1}(\sqrt{2}-1) = \frac{\pi}{8} \blacksquare \end{aligned}$$

92. 제 2종 오일러 치환 : $\sqrt{-x^2+x+2} = xt + \sqrt{2}$, $x = \frac{1-2\sqrt{2}t}{t^2+1}$, $dx = \frac{2\sqrt{2}t^2-2t-2\sqrt{2}}{(t^2+1)^2} dt$

$$\begin{aligned} \int_1^{\frac{25}{73}} \frac{1}{x\sqrt{-x^2+x+2}} dx &= \int_0^{\frac{4\sqrt{2}}{25}} \frac{t^2+1}{1-2\sqrt{2}t} \cdot \frac{t^2+1}{-\sqrt{2}t^2+t+\sqrt{2}} \cdot \frac{2\sqrt{2}t^2-2t-2\sqrt{2}}{(t^2+1)^2} dt \\ &= \int_0^{\frac{4\sqrt{2}}{25}} \frac{-2}{-2\sqrt{2}t+1} dt = \frac{1}{\sqrt{2}} \int_0^{\frac{4\sqrt{2}}{25}} \frac{-2\sqrt{2}}{-2\sqrt{2}t+1} dt = \left[\frac{1}{\sqrt{2}} \ln|2\sqrt{2}t-1| \right]_0^{\frac{4\sqrt{2}}{25}} \\ &= \sqrt{2} \ln \frac{3}{5} \blacksquare \end{aligned}$$

93. 제 3종 오일러 치환 : $\sqrt{-(x-2)(x-1)} = (x-2)t$, $x = \frac{-2t^2-1}{-t^2-1}$, $dx = \frac{2t}{(-t^2-1)^2} dt$

$$\begin{aligned} \int_1^2 \frac{x^2}{\sqrt{-x^2+3x-2}} dx &= \int_0^{-\infty} \left(\frac{-2t^2-1}{-t^2-1} \right)^2 \cdot \frac{-t^2-1}{t} \cdot \frac{2t}{(-t^2-1)^2} dt \\ &= -2 \int_0^{-\infty} \frac{(2t^2+1)^2}{(t^2+1)^3} dt = -2 \int_0^{-\infty} \left(\frac{4}{t^2+1} - \frac{4}{(t^2+1)^2} + \frac{1}{(t^2+1)^3} \right) dt \\ &= \left[-8 \tan^{-1}t + 8I(t) - 2J(t) \right]_0^{-\infty} = \left[-\frac{19}{4} \tan^{-1}t + \frac{t(13t^2+11)}{4(t^2+1)^2} \right]_0^{-\infty} = \frac{19}{8} \pi \blacksquare \end{aligned}$$

$$\begin{aligned} \ast \quad t = \tan u, \quad dt = \sec^2 u du, \quad I(t) &= \int_0^{-\infty} \frac{1}{(t^2+1)^2} dt = \int_0^{-\frac{\pi}{2}} \cos^2 u du \\ &= \frac{1}{2} \int_0^{-\frac{\pi}{2}} (1 + \cos 2u) du = \left[\frac{1}{2} u + \frac{1}{4} \sin 2u \right]_0^{-\frac{\pi}{2}} = \left[\frac{1}{2} \tan^{-1} t + \frac{t}{2(1+t^2)} \right]_0^{-\infty} \end{aligned}$$

$$\begin{aligned} \ast\ast \quad t = \tan u, \quad dt = \sec^2 u du, \quad J(t) &= \int_0^{-\infty} \frac{1}{(t^2+1)^3} dt = \int_0^{-\frac{\pi}{2}} \cos^4 u du \\ &= \frac{1}{4} \int_0^{-\frac{\pi}{2}} (1 + \cos 2u)^2 du = \frac{1}{4} \int_0^{-\frac{\pi}{2}} (1 + 2\cos 2u + \cos^2 2u) du \\ &= \frac{1}{4} \int_0^{-\frac{\pi}{2}} \left(1 + 2\cos 2u + \frac{1 + \cos 4u}{2} \right) du = \frac{1}{8} \int_0^{-\frac{\pi}{2}} (3 + 4\cos 2u + \cos 4u) du \\ &= \left[\frac{3}{8} u + \frac{1}{4} \sin 2u + \frac{1}{32} \sin 4u \right]_0^{-\frac{\pi}{2}} = \left[\frac{3}{8} \tan^{-1} t + \frac{t(3t^2+5)}{8(t^2+1)^2} \right]_0^{-\infty} \end{aligned}$$

$$94. \quad t = \cos \theta, \quad dt = -\sin \theta d\theta, \quad u = \frac{t-1}{t+1}, \quad t = \frac{1+u}{1-u}, \quad dt = \frac{2}{(1-u)^2} du$$

$$\begin{aligned} &\int_0^{\frac{\pi}{4}} \frac{\sin^3(\theta/2)}{\cos(\theta/2) \cdot \sqrt{\cos^3 \theta + \cos^2 \theta + \cos \theta}} d\theta \\ &= \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{(1 - \cos \theta) \sin \theta}{(1 + \cos \theta) \sqrt{\cos^3 \theta + \cos^2 \theta + \cos \theta}} d\theta = \frac{1}{2} \int_1^{\frac{1}{\sqrt{2}}} \frac{t-1}{(t+1) \sqrt{t^3+t^2+t}} dt \\ &= \frac{1}{2} \int_0^{2\sqrt{2}-3} u \cdot \frac{\sqrt{(1-u)^3}}{\sqrt{(1+u)^3 + (1+u)^2(1-u) + (1+u)(1-u)^2}} \cdot \frac{2}{(1-u)^2} du \\ &= \int_0^{2\sqrt{2}-3} \frac{u}{\sqrt{3-2u^2-u^4}} du = \frac{1}{2} \int_0^{2\sqrt{2}-3} \frac{2u}{\sqrt{4-(u^2+1)^2}} du \\ &= \left[\frac{1}{2} \sin^{-1} \left(\frac{u^2+1}{2} \right) \right]_0^{2\sqrt{2}-3} = \frac{1}{2} \sin^{-1}(9-6\sqrt{2}) - \frac{\pi}{12} \quad \blacksquare \end{aligned}$$

$$95. \quad \int_0^{\frac{\pi}{6}} \frac{\tan^4 \theta}{1 - \tan^2 \theta} d\theta = \int_0^{\frac{\pi}{6}} \frac{\tan^4 \theta - 1}{1 - \tan^2 \theta} d\theta + \int_0^{\frac{\pi}{6}} \frac{1}{1 - \tan^2 \theta} d\theta$$

$$\begin{aligned}
&= -\int_0^{\frac{\pi}{6}} (\tan^2\theta + 1)d\theta + \int_0^{\frac{\pi}{6}} \frac{\cos^2\theta}{\cos^2\theta - \sin^2\theta}d\theta = -\int_0^{\frac{\pi}{6}} \sec^2\theta d\theta + \int_0^{\frac{\pi}{6}} \frac{1 + \cos 2\theta}{2\cos 2\theta}d\theta \\
&= [-\tan\theta]_0^{\frac{\pi}{6}} + \int_0^{\frac{\pi}{6}} \left(\frac{1}{2} \sec 2\theta + \frac{1}{2}\right)d\theta = \left[-\tan\theta + \frac{1}{4} \ln |\sec 2\theta + \tan 2\theta| + \frac{1}{2}\theta\right]_0^{\frac{\pi}{6}} \\
&= -\frac{1}{\sqrt{3}} + \frac{\pi}{12} + \frac{1}{4} \ln(2 + \sqrt{3}) \blacksquare
\end{aligned}$$

$$\begin{aligned}
96. \int_{-\infty}^{\infty} \frac{x^2}{x^4 + 1} dx &= \int_{-\infty}^{\infty} \frac{1}{\left(x - \frac{1}{x}\right)^2 + 2} dx = \int_{-\infty}^{\infty} \frac{1}{x^2 + 2} dx \quad (\because \text{Glasser's Master Theorem}) \\
&= \left[\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) \right]_{-\infty}^{\infty} = \frac{\pi}{\sqrt{2}} \blacksquare
\end{aligned}$$

$$\begin{aligned}
97. \int_0^{\infty} \exp\left(-x^2 - \frac{1}{x^2}\right) dx &= \int_0^{\infty} \exp\left[-\left(x - \frac{1}{x}\right)^2 - 2\right] dx = \frac{1}{2e^2} \int_{-\infty}^{\infty} \exp\left[-\left(x - \frac{1}{x}\right)^2\right] dx \\
&= \frac{1}{2e^2} \int_{-\infty}^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2e^2} \blacksquare \quad (\because \text{Glasser's Master Theorem})
\end{aligned}$$

$$\begin{aligned}
98. \int_0^1 \frac{1 - x^{99}}{(1+x)(1+x^{100})} dx &= \int_0^1 \frac{(1+x^{100}) - x^{99}(1+x)}{(1+x)(1+x^{100})} dx = \int_0^1 \left(\frac{1}{1+x} - \frac{x^{99}}{1+x^{100}}\right) dx \\
&= \left[\ln|1+x| - \frac{1}{100} \ln|1+x^{100}| \right]_0^1 = \frac{99}{100} \ln 2 \blacksquare
\end{aligned}$$

$$\begin{aligned}
99. \int_{-\infty}^{\infty} \exp\left(-\frac{(x^2 - 13x - 1)^2}{611x^2}\right) dx &= \int_{-\infty}^{\infty} \exp\left(-\frac{(x - x^{-1} - 13)^2}{611}\right) dx \\
&= \int_{-\infty}^{\infty} \exp\left(-\frac{(x-13)^2}{611}\right) dx \quad (\because \text{Glasser's Master Theorem}) \\
&= \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{611}\right) dx \quad (x \mapsto x+13, dx \mapsto dx) \\
&= \int_{-\infty}^{\infty} \exp(-u^2) \cdot \sqrt{611} du \quad (u = \frac{x}{\sqrt{611}}, dx = \sqrt{611} du)
\end{aligned}$$

$$= \sqrt{611\pi} \blacksquare$$

100. $t = \tan x, dt = \sec^2 x dx$

$$\int_0^{\frac{\pi}{2}} \sqrt[n]{\tan x} dx = \int_0^{\infty} \frac{t^{\frac{1}{n}}}{1+t^2} dt, \quad \int_0^{\infty} \frac{x^a}{1+x^2} dx = \frac{\pi}{2} \sec\left(\frac{a\pi}{2}\right) \text{임을 이용하자.}$$

sol 1) $t = e^x, dt = e^x dx$

$$\int_0^{\infty} \frac{x^a}{1+x^2} dx = \int_{-\infty}^{\infty} \frac{e^{(a+1)t}}{1+e^{2t}} dt = \int_{-\infty}^0 \frac{e^{(a+1)t}}{1+e^{2t}} dt + \int_0^{\infty} \frac{e^{(a-1)t}}{1+e^{-2t}} dt$$

한편 $\frac{e^{(a+1)t}}{1+e^{2t}}, \frac{e^{(a-1)t}}{1+e^{-2t}}$ 는 공비가 각각 $(-e^{2t}), (-e^{-2t})$ 이고 초항이 각각 $e^{(a+1)t}, e^{(a-1)t}$ 인 등비급수의 합이므로 이를 역으로 전개할 수 있다.

$$\begin{aligned} & \int_{-\infty}^0 \frac{e^{(a+1)t}}{1+e^{2t}} dt + \int_0^{\infty} \frac{e^{(a-1)t}}{1+e^{-2t}} dt \\ &= \int_{-\infty}^0 \sum_{n=0}^{\infty} e^{(a+1)t} \cdot (-e^{2t})^n dt + \int_0^{\infty} \sum_{n=0}^{\infty} e^{(a-1)t} \cdot (-e^{-2t})^n dt \\ &= \sum_{n=0}^{\infty} (-1)^n \left(\int_{-\infty}^0 e^{(2n+1+a)t} dt + \int_0^{\infty} e^{-(2n+1-a)t} dt \right) \\ &= \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2n+1+a} + \frac{1}{2n+1-a} \right) = 2 \sum_{n=0}^{\infty} (-1)^n \left(\frac{2n+1}{(2n+1)^2 - a^2} \right) \dots \quad [\#] \end{aligned}$$

한편 $\sec(z)$ 의 Mittag-Leffler 전개식은

$$\sec(z) = \pi \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{(\pi/2)^2 (2n+1)^2 - z^2}$$

이므로 (참고 : <https://cdlbb.github.io/WandW/CMA07-3-FactorTheoremMN.html#7.4theexpansionofaclassoffunctionsinrationalfractions.>)

$z = \frac{a\pi}{2}$ 를 대입하고 $\frac{\pi}{2}$ 를 곱하면

$$\frac{\pi}{2} \sec\left(\frac{a\pi}{2}\right) = \frac{\pi^2}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{(\pi/2)^2 (2n+1)^2 - (a\pi/2)^2} = 2 \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{(2n+1)^2 - a^2}$$

을 얻고 이는 [#]의 전개식과 일치한다.

$$\therefore \int_0^{\infty} \frac{x^a}{1+x^2} dx = \frac{\pi}{2} \sec\left(\frac{a\pi}{2}\right) \blacksquare$$

sol 2) $t = x^2, dt = 2x dx$

$$\begin{aligned} \int_0^{\infty} \frac{x^a}{1+x^2} dx &= \frac{1}{2} \int_0^{\infty} \frac{t^{(a-1)/2}}{1+t} dt = \frac{1}{2} B\left(\frac{a+1}{2}, 1 - \frac{a+1}{2}\right) \\ &= \frac{1}{2} \Gamma\left(\frac{a+1}{2}\right) \Gamma\left(1 - \frac{a+1}{2}\right) = \frac{\pi}{2} \csc\left(\frac{(a+1)\pi}{2}\right) = \frac{\pi}{2} \sec\left(\frac{a\pi}{2}\right) \blacksquare \end{aligned}$$

※ 베타함수 $B(a, b)$ 는 다음과 같이 정의되는 이변수 함수이다.

$$B(a, b) := \int_0^{\infty} \frac{z^{a-1}}{1+z^{a+b}} dz$$

또한 이는 다음과 같이 감마함수의 비로 표현할 수 있다.

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

감마함수 $\Gamma(z)$ 는 다음과 같이 정의되며, 이는 계승 함수($n!$)의 정의역을 실수부가 양수인 복소수 전체로 확장(음의 정수는 제외)한 것이다. (29번 문제의 1번 솔루션은 이 감마함수의 정의를 이용한 것으로, 자연수 n 에 대하여 $\Gamma(n) = (n-1)!$ 이다.) 감마함수는 계승함수를 해석적으로 확장한 것이므로 팩토리얼의 성질이 그대로 성립하여, $\Gamma(x+1) = x\Gamma(x)$ 이고 $\Gamma(1) = 1$ 이다.)

$$\Gamma(z) := \int_0^{\infty} t^{z-1} e^{-t} dt \quad (Re(z) > 0)$$

감마함수에 대하여 다음과 같은 오일러 반사 공식이 성립하며, 상술한 2번 솔루션의 마지막 단계에서 이 공식이 사용되었다.

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$$

위 공식은 Residue Theorem을 사용하여 직접 복소적분을 하거나 감마함수의 정의식을 대입하여 증명할 수 있으며, 본문에서는 복소적분을 사용하지 않는다는 철학을 유지하기 위해 다른 증명법을 제시한다. 복소적분을 포함한 다른 증명법에 관심이 있는 독자는 아래 링크의 증명들을 참고하기 바란다.

<https://math.stackexchange.com/questions/714482/prove-that-gamma-p-times-gamma-1-p-frac-pi-sin-p-pi-forall-p-in?noredirect=1&lq=1>

pf) 감마함수의 바이어슈트라스 무한곱 꼴을 생각하자. (감마함수는 실제로 여러 가지 형태로 정의되고 이들은 모두 동치이다. 상술한 정의는 적분 형태이다.)

$$\frac{1}{\Gamma(p)} = pe^{\gamma p} \prod_{n=1}^{\infty} \left(1 + \frac{p}{n}\right) e^{-\frac{p}{n}}$$

$$\frac{1}{\Gamma(p)} \cdot \frac{1}{\Gamma(-p)} = pe^{\gamma p} \prod_{n=1}^{\infty} \left(1 + \frac{p}{n}\right) e^{-\frac{p}{n}} \cdot (-p)e^{-\gamma p} \prod_{n=1}^{\infty} \left(1 - \frac{p}{n}\right) e^{\frac{p}{n}} = -p^2 \prod_{n=1}^{\infty} \left(1 - \frac{p^2}{n^2}\right)$$

한편 감마함수의 성질에 의해 $\Gamma(1-p) = -p\Gamma(-p)$ 이므로

$$\frac{1}{\Gamma(1-p)\Gamma(p)} = p \prod_{n=1}^{\infty} \left(1 - \frac{p^2}{n^2}\right)$$

이고 바이어슈트라스 분해 정리에 의해

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2 \pi^2}\right)$$

임을 이용하면

$$\frac{1}{\Gamma(1-p)\Gamma(p)} = p \prod_{n=1}^{\infty} \left(1 - \frac{p^2}{n^2}\right) = \frac{\sin(\pi p)}{\pi}$$

연고 증명이 완료되었다. ■

$$\therefore \int_0^{\frac{\pi}{2}} \sqrt[n]{\tan x} dx = \int_0^{\infty} \frac{t^{\frac{1}{n}}}{1+t^2} dt = \frac{\pi}{2} \sec\left(\frac{\pi}{2n}\right) \quad \blacksquare \quad (2 \leq n \in \mathbb{N})$$