

정적분 100題 해설

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$$1. I = \int_0^{\pi} \frac{x \sin x}{3 + \sin^2 x} dx = \int_0^{\pi} \frac{(\pi - x) \sin x}{3 + \sin^2 x} dx \quad (x \mapsto \pi - x)$$

$$2I = \int_0^{\pi} \left(\frac{x \sin x}{3 + \sin^2 x} + \frac{(\pi - x) \sin x}{3 + \sin^2 x} \right) dx = \int_0^{\pi} \frac{\pi \sin x}{3 + \sin^2 x} dx = \int_0^{\pi} \frac{\pi \sin x}{4 - \cos^2 x} dx$$

$$= \int_1^{-1} \frac{\pi}{u^2 - 4} du \quad (u = \cos x, du = -\sin x dx)$$

$$= \pi \left[\frac{1}{4} \ln \left| \frac{u-2}{u+2} \right| \right]_1^{-1} = \frac{\pi}{2} \ln 3$$

$$\therefore I = \frac{\pi}{4} \ln 3 \quad \blacksquare$$

$$2. I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx$$

$$\frac{\pi}{2} - 2I = \int_0^{\frac{\pi}{2}} dx - \int_0^{\frac{\pi}{2}} \frac{2 \sin x}{\sin x + \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{\sin x + \cos x} dx = [\ln |\sin x + \cos x|]_0^{\frac{\pi}{2}} = 0$$

$$\therefore I = \frac{\pi}{4} \quad \blacksquare$$

$$3. I = \int_0^{\frac{\pi}{2}} \frac{\cos^{\sqrt{5}} x}{\sin^{\sqrt{5}} x + \cos^{\sqrt{5}} x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin^{\sqrt{5}} x}{\sin^{\sqrt{5}} x + \cos^{\sqrt{5}} x} dx \quad (x \mapsto \frac{\pi}{2} - x)$$

$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{\cos^{\sqrt{5}} x}{\sin^{\sqrt{5}} x + \cos^{\sqrt{5}} x} + \frac{\sin^{\sqrt{5}} x}{\sin^{\sqrt{5}} x + \cos^{\sqrt{5}} x} \right) dx = \frac{\pi}{2}$$

$$\therefore I = \frac{\pi}{4} \quad \blacksquare$$

$$\begin{aligned}
4. \int_0^{\frac{\pi}{4}} \frac{\sin x}{\sin x + \cos x} dx &= \int_0^{\frac{\pi}{4}} \frac{\sin\left(\frac{\pi}{4} - x\right)}{\sin\left(\frac{\pi}{4} - x\right) + \cos\left(\frac{\pi}{4} - x\right)} dx \quad (x \mapsto \frac{\pi}{4} - x) \\
&= \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{\cos x - \sin x}{\cos x} dx = \frac{1}{2} \int_0^{\frac{\pi}{4}} (1 - \tan x) dx = \frac{1}{2} [x + \ln |\cos x|]_0^{\frac{\pi}{4}} = \frac{\pi}{8} - \frac{1}{4} \ln 2 \quad \blacksquare
\end{aligned}$$

$$5. t = \sqrt[3]{x}, \quad dx = 3t^2 dt$$

$$\begin{aligned}
\int_0^8 \frac{x^3 - 2x + 1}{\sqrt[3]{x}} dx &= \int_0^2 3t(t^9 - 2t^3 + 1) dt = \left[\frac{3}{11} t^{11} - \frac{6}{5} t^5 + \frac{3}{2} t^2 \right]_0^2 \\
&= \left[\frac{3}{11} x^{\frac{11}{3}} - \frac{6}{5} x^{\frac{5}{3}} + \frac{3}{2} x^{\frac{2}{3}} \right]_0^8 = \frac{28938}{55} \quad \blacksquare
\end{aligned}$$

$$6. I = \int_{-3}^3 \frac{2x^2}{2^x + 1} dx = \int_{-3}^3 \frac{2x^2}{2^{-x} + 1} dx = \int_{-3}^3 \frac{2^x \cdot 2x^2}{2^x + 1} dx \quad (x \mapsto -x)$$

$$2I = \int_{-3}^3 \left(\frac{2x^2}{2^x + 1} + \frac{2^x \cdot 2x^2}{2^x + 1} \right) dx = \int_{-3}^3 2x^2 dx = \left[\frac{2}{3} x^3 \right]_{-3}^3 = 36$$

$$\therefore I = \int_{-3}^3 \frac{2x^2}{2^x + 1} dx = 18 \quad \blacksquare$$

$$7. I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{e^{1/x} + 1} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{e^{-1/x} + 1} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{1/x} \cdot \cos x}{e^{1/x} + 1} dx \quad (x \mapsto -x)$$

$$2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\cos x}{e^{1/x} + 1} + \frac{e^{1/x} \cdot \cos x}{e^{1/x} + 1} \right) dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x dx = 2$$

$$\therefore I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{e^{1/x} + 1} dx = 1 \quad \blacksquare$$

$$8. \int_0^{\frac{\pi}{2}} \sin 3x \cos 4x dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} (\sin 7x - \sin x) dx = \left[\frac{1}{2} \cos x - \frac{1}{14} \cos 7x \right]_0^{\frac{\pi}{2}} = -\frac{3}{7} \quad \blacksquare$$

$$9. [1] \quad t^3 = \tan x, \quad x = \tan^{-1}(t^3), \quad dx = \frac{3t^2}{1+t^6} dt$$

$$[2] \quad a = t^2, \quad da = 2t dt, \quad a = \tan^{2/3} x$$

$$\int_0^{\pi/2} \sqrt[3]{\tan x} dx = \int_0^\infty \frac{3t^3}{1+t^6} dt \quad \dots [1]$$

$$= \frac{3}{2} \int_0^\infty \frac{a}{1+a^3} da \quad \dots [2]$$

$$= \frac{3}{2} \int_0^\infty \left(-\frac{1}{3(a+1)} + \frac{a+1}{3(a^2-a+1)} \right) da$$

$$= \frac{1}{2} \int_0^\infty \left(\frac{2a-1}{2(a^2-a+1)} + \frac{3}{2(a^2-a+1)} - \frac{1}{a+1} \right) da$$

$$= \left[\frac{1}{4} \ln \frac{a^2-a+1}{(a+1)^2} + \frac{\sqrt{3}}{2} \tan^{-1} \left(\frac{2}{\sqrt{3}} \left(a - \frac{1}{2} \right) \right) \right]_0^\infty$$

$$= \left[\frac{1}{4} \ln \frac{\tan^{4/3} x - \tan^{2/3} x + 1}{(\tan^{2/3} x + 1)^2} + \frac{\sqrt{3}}{2} \tan^{-1} \left(\frac{2}{\sqrt{3}} \left(\tan^{2/3} x - \frac{1}{2} \right) \right) \right]_0^{\pi/2}$$

$$= \lim_{x \rightarrow \pi/2} \left\{ \frac{1}{4} \ln \frac{\tan^{4/3} x - \tan^{2/3} x + 1}{(\tan^{2/3} x + 1)^2} + \frac{\sqrt{3}}{2} \tan^{-1} \left(\frac{2}{\sqrt{3}} \left(\tan^{2/3} x - \frac{1}{2} \right) \right) \right\} - \frac{\sqrt{3}}{2} \tan^{-1} \left(-\frac{1}{\sqrt{3}} \right)$$

$$= \lim_{t \rightarrow \infty} \left\{ \frac{1}{4} \ln \frac{t^{4/3} - t^{2/3} + 1}{(t^{2/3} + 1)^2} + \frac{\sqrt{3}}{2} \tan^{-1} \left(\frac{2}{\sqrt{3}} \left(t^{2/3} - \frac{1}{2} \right) \right) \right\} + \frac{\sqrt{3}}{12} \pi$$

$$= \frac{1}{4} \ln 1 + \frac{\sqrt{3}}{4} \pi + \frac{\sqrt{3}}{12} \pi = \frac{\pi}{\sqrt{3}} \quad \blacksquare$$

$$10. [1] \quad t^3 = \tan x, \quad x = \tan^{-1}(t^3), \quad dx = \frac{3t^2}{1+t^6} dt$$

$$[2] \quad a = t^2, \quad da = 2t dt, \quad a = \tan^{2/3} x$$

$$\int_0^{\pi/4} \sqrt[3]{\tan x} dx = \int_0^1 \frac{3t^3}{1+t^6} dt \quad \dots [1]$$

$$= \frac{3}{2} \int_0^1 \frac{a}{1+a^3} da \quad \dots [2]$$

$$\begin{aligned}
&= \frac{3}{2} \int_0^1 \left(-\frac{1}{3(a+1)} + \frac{a+1}{3(a^2-a+1)} \right) da \\
&= \frac{1}{2} \int_0^1 \left(\frac{2a-1}{2(a^2-a+1)} + \frac{3}{2(a^2-a+1)} - \frac{1}{a+1} \right) da \\
&= \left[\frac{1}{4} \ln \frac{a^2-a+1}{(a+1)^2} + \frac{\sqrt{3}}{2} \tan^{-1} \left(\frac{2}{\sqrt{3}} \left(a - \frac{1}{2} \right) \right) \right]_0^1 \\
&= \left[\frac{1}{4} \ln \frac{\tan^{4/3} x - \tan^{2/3} x + 1}{(\tan^{2/3} x + 1)^2} + \frac{\sqrt{3}}{2} \tan^{-1} \left(\frac{2}{\sqrt{3}} \left(\tan^{2/3} x - \frac{1}{2} \right) \right) \right]_0^{\frac{\pi}{4}} \\
&= \left(-\frac{1}{2} \ln 2 + \frac{\sqrt{3}}{12} \pi \right) - \left(0 - \frac{\sqrt{3}}{12} \pi \right) = \frac{\sqrt{3}}{6} \pi - \frac{1}{2} \ln 2 \blacksquare
\end{aligned}$$

$$\begin{aligned}
11. \quad \int_0^1 \frac{1}{x^x} dx &= \int_0^1 e^{-x \ln x} dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-x \ln x)^n}{n!} dx = \sum_{n=0}^{\infty} \int_0^1 \frac{(-x \ln x)^n}{n!} dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^n (\ln x)^n dx
\end{aligned}$$

한편

$$\begin{aligned}
\int_0^1 x^n (\ln x)^n dx &= \left[\frac{1}{n+1} x^{n+1} (\ln x)^n \right]_0^1 - \int_0^1 \frac{1}{n+1} x^{n+1} \cdot \frac{n(\ln x)^{n-1}}{x} dx \\
&= -\frac{n}{n+1} \int_0^1 x^n (\ln x)^{n-1} dx = (-1)^2 \frac{n(n-1)}{(n+1)^2} \int_0^1 x^n (\ln x)^{n-2} dx \\
&= (-1)^3 \frac{n(n-1)(n-2)}{(n+1)^3} \int_0^1 x^n (\ln x)^{n-3} dx = \dots \\
&= (-1)^n \frac{n!}{(n+1)^n} \int_0^1 x^n (\ln x)^{n-n} dx = (-1)^n \frac{n!}{(n+1)^{n+1}}
\end{aligned}$$

이므로

$$\therefore \int_0^1 \frac{1}{x^x} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^n (\ln x)^n dx = \sum_{n=0}^{\infty} (n+1)^{-(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n^n} \blacksquare$$

$$\begin{aligned}
12. \quad I &= \int_0^1 \frac{1}{1 + \left(1 - \frac{1}{x}\right)^{2022}} dx = \int_0^1 \frac{1}{1 + \left(\frac{x-1}{x}\right)^{2022}} dx = \int_0^1 \frac{x^{2022}}{x^{2022} + (x-1)^{2022}} dx \\
&= \int_0^1 \frac{(1-x)^{2022}}{x^{2022} + (x-1)^{2022}} dx \quad (x \mapsto 1-x) \\
&= \frac{1}{2} \int_0^1 \left(\frac{x^{2022}}{x^{2022} + (1-x)^{2022}} + \frac{(1-x)^{2022}}{x^{2022} + (1-x)^{2022}} \right) dx = \frac{1}{2} \blacksquare
\end{aligned}$$

$$13. \quad I = \int_{-\infty}^{\infty} e^{-x^2} dx = 2 \int_0^{\infty} e^{-x^2} dx \quad (\text{Gaussian Integral, 가우스 적분})$$

$$\begin{aligned}
I^2 &= 4 \int_0^{\infty} e^{-x^2} dx \cdot \int_0^{\infty} e^{-y^2} dy = 4 \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy \\
&= 4 \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-r^2} r dr d\theta \quad (r = x^2 + y^2, \theta = \tan^{-1}\left(\frac{y}{x}\right), dx dy = r dr d\theta) \\
&= 4 \int_0^{\frac{\pi}{2}} d\theta \cdot \int_0^{\infty} r e^{-r^2} dr = 2\pi \left[-\frac{1}{2} e^{-r^2} \right]_0^{\infty} = \pi \\
\therefore I &= \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \blacksquare
\end{aligned}$$

$$14. \quad z = \frac{x-\mu}{\sigma}, \quad dz = \frac{dx}{\sigma} \quad (N(\mu, \sigma^2) \text{을 따르는 정규분포의 확률밀도함수})$$

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} dy \quad (z = \sqrt{2}y, dz = \sqrt{2}dy) \\
&= 1 \blacksquare
\end{aligned}$$

$$15. \quad \int_0^{\infty} 4\pi r^2 |\psi(r)|^2 dr = \frac{4}{a_0^3} \int_0^{\infty} r^2 e^{-\frac{2r}{a_0}} dr = \frac{4}{a_0^3} \left\{ \left[-\frac{a_0}{2} r^2 e^{-\frac{2r}{a_0}} \right]_0^{\infty} + \frac{a_0}{2} \int_0^{\infty} 2re^{-\frac{2r}{a_0}} dr \right\}$$

$$\begin{aligned}
&= \frac{4}{a_0^3} \left\{ 0 + a_0 \left[-\frac{a_0}{2} r e^{-\frac{2r}{a_0}} \right]_0^\infty + \frac{a_0^2}{2} \int_0^\infty e^{-\frac{2r}{a_0}} dr \right\} \\
&= \frac{4}{a_0^3} \left\{ \frac{a_0^2}{2} \left[-\frac{a_0}{2} e^{-\frac{2r}{a_0}} \right]_0^\infty \right\} = \frac{4}{a_0^3} \left(\frac{a_0^3}{4} \right) = 1 \blacksquare
\end{aligned}$$

$\psi(r)$ 은 거리 r 에 따른 수소원자의 바닥상태 파동함수이다. 위 적분은 수소원자의 전구간에서 전자가 발견될 확률이 1임을 의미한다.

$$\begin{aligned}
16. \int_{\frac{a_0}{2}}^\infty 4\pi r^2 |\psi(r)|^2 dr &= \frac{4}{a_0^3} \int_{\frac{a_0}{2}}^\infty r^2 e^{-\frac{2r}{a_0}} dr = \frac{4}{a_0^3} \left\{ \left[-\frac{a_0}{2} r^2 e^{-\frac{2r}{a_0}} \right]_{\frac{a_0}{2}}^\infty + \frac{a_0}{2} \int_{\frac{a_0}{2}}^\infty 2r e^{-\frac{2r}{a_0}} dr \right\} \\
&= \frac{4}{a_0^3} \left\{ \frac{a_0^3}{8e} + a_0 \left[-\frac{a_0}{2} r e^{-\frac{2r}{a_0}} \right]_{\frac{a_0}{2}}^\infty + \frac{a_0^2}{2} \int_{\frac{a_0}{2}}^\infty e^{-\frac{2r}{a_0}} dr \right\} \\
&= \frac{4}{a_0^3} \left\{ \frac{3a_0^3}{8e} + \frac{a_0^2}{2} \left[-\frac{a_0}{2} e^{-\frac{2r}{a_0}} \right]_{\frac{a_0}{2}}^\infty \right\} = \frac{4}{a_0^3} \left(\frac{5a_0^3}{8e} \right) = \frac{5}{2e} \simeq 91.97\% \blacksquare
\end{aligned}$$

$\psi(r)$ 은 거리 r 에 따른 수소원자의 바닥상태 파동함수이다. 위 적분은 수소원자에서 전자가 핵으로부터 $\frac{a_0}{2}$ 보다 멀리서 발견될 확률이 약 91.97%임을 의미한다. (a_0 는 보어 반지름이다.) [2021 물리인증 1급 기출]

$$\begin{aligned}
17. W_n &:= \int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \sin^{n-1} x \cdot \sin x dx \\
&= [-\cos x \cdot \sin^{n-1} x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (n-1) \sin^{n-2} x \cos x \cdot (-\cos x) dx \\
&= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x dx = (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x (1 - \sin^2 x) dx \\
&= (n-1)(W_{n-2} - W_n) \\
n W_n &= (n-1) W_{n-2}, \quad W_n = \frac{n-1}{n} W_{n-2} \quad (n \geq 2)
\end{aligned}$$

한편

$$W_0 = \int_0^{\frac{\pi}{2}} 1 dx = \frac{\pi}{2}, \quad W_1 = \int_0^{\frac{\pi}{2}} \sin x dx = 1$$

이므로

$$W_n = \frac{n-1}{n} W_{n-2} = \frac{(n-1)(n-3)}{n(n-2)} W_{n-4} = \dots$$

$$= \begin{cases} \frac{(n-1)(n-3)\cdots 1}{n(n-2)\cdots 2} \cdot W_0 & (n=2m) \\ \frac{(n-1)(n-3)\cdots 2}{n(n-2)\cdots 1} \cdot W_1 & (n=2m+1) \end{cases}$$

$$\therefore W_n = \frac{(n-1)!!}{n!!} \cdot \left(\frac{\pi}{2}\right)^{\frac{1+(-1)^n}{2}} \quad (n > 0), \quad W_0 = \frac{\pi}{2} \quad (\text{Wallis' Formula})$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^{10} x dx = W_{10} = \frac{9 \cdot 7 \cdot 5 \cdot 3 \cdot 1}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{63\pi}{512} \blacksquare$$

18. $I(a) := \int_0^1 \frac{x^a - 1}{\ln x} dx$, 파인만 적분 테크닉을 적용하면

$$\frac{d}{da} I(a) = \int_0^1 \frac{\partial}{\partial a} \left(\frac{x^a - 1}{\ln x} \right) dx = \int_0^1 x^a dx = \left[\frac{1}{a+1} x^{a+1} \right]_0^1 = \frac{1}{a+1}$$

$$I(a) = \int \frac{1}{a+1} da = \ln |a+1| + C$$

이때 $I(0) = \int_0^1 0 dx = 0$ 이므로 $C = 0$ 이다.

$$\therefore I(2022) = \int_0^1 \frac{x^{2022} - 1}{\ln x} dx = \ln 2023 \blacksquare$$

19. $(1+x)^{-1}$ 의 이항급수를 구하고 양변을 적분하면

$$(1+x)^{-1} = \sum_{n=0}^{\infty} \binom{-1}{n} \cdot x^n = \sum_{n=0}^{\infty} \prod_{i=0}^{n-1} (-1-i) \cdot \frac{x^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \cdot x^n \quad (|x| < 1)$$

$$\int (1+x)^{-1} dx = \ln|1+x| = \int \sum_{n=0}^{\infty} (-1)^n x^n dx + C = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + C$$

$x=0$ 을 대입하면 $0=0+C$ 에서 $C=0$ 이다.

$$\therefore \int_0^1 \frac{\ln x}{x-1} dx = \int_{-1}^0 \frac{\ln(t+1)}{t} dt = \int_{-1}^0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} t^n dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} t^{n+1} \right]_{-1}^0$$

$$= \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} 0^{n+1} \right] - \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} (-1)^{n+1} \right] = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$$

$$= \zeta(2) = \frac{\pi^2}{6} \quad (\because \text{바젤 문제에 의해 } \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6})$$

$$20. I(\alpha) := \int_0^{\infty} \frac{\sin x}{x} e^{-\alpha x} dx, \quad \frac{d}{d\alpha} I(\alpha) = \int_0^{\infty} \frac{\partial}{\partial \alpha} \frac{\sin x}{x} e^{-\alpha x} dx = - \int_0^{\infty} e^{-\alpha x} \sin x dx$$

$$= - [-e^{-\alpha x} \cos x]_0^{\infty} + \int_0^{\infty} \alpha e^{-\alpha x} \cos x dx = -1 + [\alpha e^{-\alpha x} \sin x]_0^{\infty} + \int_0^{\infty} \alpha^2 e^{-\alpha x} \sin x dx$$

$$= -1 - \alpha^2 I'(\alpha), \quad I'(\alpha) = - \frac{1}{1+\alpha^2}$$

$$I(\alpha) = \int I'(\alpha) d\alpha = C - \tan^{-1} \alpha \quad (C \in \mathbb{R}), \quad 0 = \lim_{\alpha \rightarrow \infty} I(\alpha) = C - \frac{\pi}{2} \text{에서 } C = \frac{\pi}{2} \text{이다.}$$

$$\therefore \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = 2 \int_0^{\infty} \frac{\sin x}{x} dx = 2I(0) = \pi \quad \blacksquare$$

$$21. \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = \left[-\frac{\sin^2 x}{x} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{2 \sin x \cos x}{x} dx = \int_{-\infty}^{\infty} \frac{\sin 2x}{x} dx$$

$$= \int_{-\infty}^{\infty} \frac{\sin t}{t} dt \quad (t=2x, dt=2dx)$$

$$I(\alpha) := \int_0^{\infty} \frac{\sin x}{x} e^{-\alpha x} dx, \quad \frac{d}{d\alpha} I(\alpha) = \int_0^{\infty} \frac{\partial}{\partial \alpha} \frac{\sin x}{x} e^{-\alpha x} dx = - \int_0^{\infty} e^{-\alpha x} \sin x dx$$

$$= - [-e^{-\alpha x} \cos x]_0^{\infty} + \int_0^{\infty} \alpha e^{-\alpha x} \cos x dx = -1 + [\alpha e^{-\alpha x} \sin x]_0^{\infty} + \int_0^{\infty} \alpha^2 e^{-\alpha x} \sin x dx$$

$$= -1 - \alpha^2 I'(\alpha), \quad I'(\alpha) = -\frac{1}{1+\alpha^2}$$

$$I(\alpha) = \int I'(\alpha) d\alpha = C - \tan^{-1} \alpha \quad (C \in \mathbb{R}), \quad 0 = \lim_{\alpha \rightarrow \infty} I(\alpha) = C - \frac{\pi}{2} \text{에서 } C = \frac{\pi}{2} \text{이다.}$$

$$\therefore \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = 2 \int_0^{\infty} \frac{\sin x}{x} dx = 2I(0) = \pi \blacksquare$$

$$22. \quad I = \int_0^1 \sqrt[3]{2x^3 - 3x^2 - x + 1} dx = \int_0^1 \sqrt[3]{2(1-x)^3 - 3(1-x)^2 - (1-x) + 1} dx \quad (x \mapsto 1-x)$$

$$= \int_0^1 -\sqrt[3]{2x^3 - 3x^2 - x + 1} dx = -I$$

$$\therefore I = \int_0^1 \sqrt[3]{2x^3 - 3x^2 - x + 1} dx = 0 \blacksquare$$

$$23. \quad x = \tan t, \quad dx = \sec^2 t dt, \quad \frac{dx}{1+x^2} = dt$$

$$I = \int_0^1 \frac{\ln(x+1)}{x^2+1} dx = \int_0^{\frac{\pi}{4}} \ln(1+\tan t) dt = \int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4}-t\right)\right) dt \quad (t \mapsto \frac{\pi}{4}-t)$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan t}{1+\tan t}\right) dt = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan t}\right) dt = \int_0^{\frac{\pi}{4}} \{\ln 2 - \ln(1+\tan t)\} dt$$

$$2I = \int_0^{\frac{\pi}{4}} \{\ln(1+\tan t) + (\ln 2 - \ln(1+\tan t))\} dt = \int_0^{\frac{\pi}{4}} \ln 2 dt = \frac{\pi}{4} \ln 2$$

$$\therefore I = \int_0^1 \frac{\ln(x+1)}{x^2+1} dx = \frac{\pi}{8} \ln 2 \blacksquare$$

$$24. \quad I = \int_0^2 \frac{\ln(x+1)}{x^2-x+1} dx = \int_1^3 \frac{\ln t}{t^2-3t+3} dt \quad (t=x+1, \quad dt=dx)$$

$$= \int_3^1 \frac{\ln(3u^{-1})}{9u^{-2}-9u^{-1}+3} \left(-\frac{3}{u^2} du\right) \quad \left(u = \frac{3}{t}, \quad du = -\frac{3}{t^2} dt, \quad dt = -\frac{3}{u^2} du\right)$$

$$\begin{aligned}
&= \int_1^3 \frac{\ln 3 - \ln u}{u^2 - 3u + 3} du = \int_1^3 \frac{\ln 3}{u^2 - 3u + 3} du - \int_1^3 \frac{\ln u}{u^2 - 3u + 3} du = \int_1^3 \frac{\ln 3}{u^2 - 3u + 3} du - I \\
&= \frac{\ln 3}{2} \int_1^3 \frac{1}{u^2 - 3u + 3} du = \frac{\ln 3}{2} \int_1^3 \frac{1}{\left(u - \frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} du = \frac{\ln 3}{2} \left[\frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2x-3}{\sqrt{3}} \right) \right]_1^3 \\
&= \frac{\ln 3}{\sqrt{3}} \left(\frac{\pi}{3} + \frac{\pi}{6} \right) = \frac{\pi \ln 3}{2\sqrt{3}} \blacksquare
\end{aligned}$$

$$\begin{aligned}
25. \quad I(\alpha) &:= \int_0^1 \frac{\ln(\alpha x^2 + 1)}{x+1} dx, \quad \frac{d}{d\alpha} I(\alpha) = \int_0^1 \frac{\partial}{\partial \alpha} \frac{\ln(\alpha x^2 + 1)}{x+1} dx = \int_0^1 \frac{x^2}{(\alpha x^2 + 1)(x+1)} dx \\
&= \int_0^1 \left(\frac{x-1}{(\alpha+1)(\alpha x^2 + 1)} + \frac{1}{(\alpha+1)(x+1)} \right) dx = \frac{1}{\alpha+1} \int_0^1 \left(\frac{x}{\alpha x^2 + 1} - \frac{1}{\alpha x^2 + 1} + \frac{1}{x+1} \right) dx \\
&= \frac{1}{\alpha+1} \left[\frac{1}{2\alpha} \ln |\alpha x^2 + 1| - \frac{1}{\sqrt{\alpha}} \tan^{-1}(\sqrt{\alpha} x) + \ln |x+1| \right]_0^1 \\
&= \frac{1}{\alpha+1} \left(\frac{\ln(\alpha+1)}{2\alpha} - \frac{\tan^{-1}(\sqrt{\alpha})}{\sqrt{\alpha}} + \ln 2 \right) \\
\therefore \int_0^1 \frac{\ln(x^2+1)}{x+1} dx &= I(1) = \frac{1}{2} \int_0^1 \frac{\ln(\alpha+1)}{\alpha(\alpha+1)} d\alpha - \int_0^1 \frac{\tan^{-1}(\alpha)}{\sqrt{\alpha}(\alpha+1)} d\alpha + \int_0^1 \frac{\ln 2}{\alpha+1} d\alpha \\
&= \frac{1}{2} \left(\int_0^1 \frac{\ln(\alpha+1)}{\alpha} d\alpha - \int_0^1 \frac{\ln(\alpha+1)}{\alpha+1} d\alpha \right) - 2 \left[\frac{1}{2} (\tan^{-1}(\alpha))^2 \right]_0^1 + \ln 2 [\ln(\alpha+1)]_0^1 \\
&= \frac{1}{2} \left(\int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^n}{n+1} d\alpha - \left[\frac{1}{2} (\ln(\alpha+1))^2 \right]_0^1 \right) - \frac{\pi^2}{16} + (\ln 2)^2 \\
&= \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \int_0^1 \alpha^n d\alpha - \frac{1}{2} (\ln 2)^2 \right) - \frac{\pi^2}{16} + (\ln 2)^2 \\
&= \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} \right) - \frac{\pi^2}{16} + \frac{3}{4} (\ln 2)^2 = \frac{1}{2} \left(\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{2n^2} \right) - \frac{\pi^2}{16} + \frac{3}{4} (\ln 2)^2 \\
&= \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{\pi^2}{16} + \frac{3}{4} (\ln 2)^2 = \frac{1}{4} \zeta(2) - \frac{\pi^2}{16} + \frac{3}{4} (\ln 2)^2
\end{aligned}$$

$$= \frac{3}{4}(\ln 2)^2 - \frac{\pi^2}{48} \quad (\because \text{바젤 문제에 의해 } \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6})$$

$$26. I = \int_0^{\frac{\pi}{2}} \ln(\cos x) dx = \int_0^{\frac{\pi}{2}} \ln(\sin x) dx \quad (x \mapsto \frac{\pi}{2} - x)$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln(\sin x \cos x) dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{2} \sin 2x\right) dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln(\sin 2x) dx - \frac{\pi}{4} \ln 2$$

$$= \frac{1}{4} \int_0^{\pi} \ln(\sin t) dt - \frac{\pi}{4} \ln 2 \quad (t = 2x, \quad dt = 2dx)$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln(\sin t) dt - \frac{\pi}{4} \ln 2 = \frac{1}{2} I - \frac{\pi}{4} \ln 2, \quad \frac{1}{2} I = -\frac{\pi}{4} \ln 2$$

$$\therefore I = \int_0^{\frac{\pi}{2}} \ln(\cos x) dx = -\frac{\pi}{2} \ln 2 \quad \blacksquare$$

$$27. \text{ sol 1) } I = \int_0^1 \frac{\ln x}{x+1} dx = \int_0^1 \frac{\ln x}{1-x} dx - \int_0^1 \frac{2x \ln x}{1-x^2} dx$$

$$= \int_0^1 \frac{\ln x}{1-x} dx - \frac{1}{2} \int_0^1 \frac{\ln t}{1-t} dt \quad (t = x^2, \quad dt = 2x dx)$$

$$= \frac{1}{2} \int_0^1 \frac{\ln t}{1-t} dt$$

$(1+x)^{-1}$ 의 이항급수를 구하고 양변을 적분하면

$$(1+x)^{-1} = \sum_{n=0}^{\infty} \binom{-1}{n} \cdot x^n = \sum_{n=0}^{\infty} \prod_{i=0}^{n-1} (-1-i) \cdot \frac{x^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \cdot x^n \quad (|x| < 1)$$

$$\int (1+x)^{-1} dx = \ln |1+x| = \int \sum_{n=0}^{\infty} (-1)^n x^n dx + C = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + C$$

$x=0$ 을 대입하면 $0=0+C$ 에서 $C=0$ 이다.

$$\int_0^1 \frac{\ln x}{x-1} dx = \int_{-1}^0 \frac{\ln(t+1)}{t} dt = \int_{-1}^0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} t^n dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} t^{n+1} \right]_{-1}^0$$

$$= \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} 0^{n+1} \right] - \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} (-1)^{n+1} \right] = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$$

$$= \zeta(2) = \frac{\pi^2}{6} \quad (\because \text{바젤 문제에 의해 } \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6})$$

$$\therefore I = \frac{1}{2} \int_0^1 \frac{\ln t}{1-t} dt = -\frac{\pi^2}{12}$$

sol 2) $\int_0^1 x^a dx = \frac{1}{a+1}$ 의 양변을 a 로 미분하면

$$\frac{d}{da} \int_0^1 x^a dx = \int_0^1 \frac{\partial}{\partial a} x^a dx = \int_0^1 x^a \ln x dx = -\frac{1}{(a+1)^2}$$

한편 $|x| < 1$ 일 때 $\sum_{a=0}^{\infty} (-1)^a x^a = \frac{1}{1+x}$ 임을 이용하면

$$\sum_{a=0}^{\infty} (-1)^a \int_0^1 x^a \ln x dx = \int_0^1 \sum_{a=0}^{\infty} (-1)^a x^a \ln x dx = -\sum_{a=0}^{\infty} \frac{(-1)^a}{(a+1)^2}$$

$$\therefore \int_0^1 \frac{\ln x}{x+1} dx = -\sum_{a=0}^{\infty} \frac{(-1)^a}{(a+1)^2} = \sum_{a=1}^{\infty} \frac{(-1)^a}{a^2} = \sum_{a=1}^{\infty} \frac{1}{2a^2} - \sum_{a=1}^{\infty} \frac{1}{a^2}$$

$$= -\frac{1}{2} \sum_{a=1}^{\infty} \frac{1}{a^2} = -\frac{1}{2} \zeta(2) = -\frac{\pi^2}{12} \quad \blacksquare$$

28. $n \in \mathbb{N} \cup \{0\}$, $a_n := \int_0^{\frac{\pi}{2}} \cos^n x \cos nx dx$

$$a_n = \int_0^{\frac{\pi}{2}} \cos^n x \{ \cos(n+1)x \cos x + \sin(n+1)x \sin x \} dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n+1} x \cos(n+1)x dx + \int_0^{\frac{\pi}{2}} \cos^n x \sin(n+1)x \sin x dx$$

$$= a_{n+1} + \left[\sin(n+1)x \cdot \frac{-1}{n+1} \cos^{n+1} x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \cos(n+1)x (-\cos^{n+1} x) dx$$

$$= 2a_{n+1}$$

$$a_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}, \quad a_1 = \int_0^{\frac{\pi}{2}} \cos^2 dx = \left[\frac{1}{2}x + \frac{1}{4} \sin 2x \right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

$$a_n = \int_0^{\frac{\pi}{2}} \cos^n x \cos nx dx = \pi \cdot \left(\frac{1}{2}\right)^{n+1}$$

$$\therefore a_{2022} = \int_0^{\frac{\pi}{2}} \cos^{2022} x \cos 2022x dx = \frac{\pi}{2^{2023}} \blacksquare$$

29. sol 1) $t = -\ln x$, $x = e^{-t}$, $dx = -e^{-t} dt$

$$\int_0^1 (\ln x)^{2022} dx = \int_0^\infty t^{2022} e^{-t} dt = \Gamma(2023) = 2022! \blacksquare$$

sol 2) $J_n := \int_0^1 (-\ln x)^n dx = [x \cdot (-\ln x)^n]_0^1 - \int_0^1 x \cdot n(-\ln x)^{n-1} \cdot \frac{-1}{x} dx$

$$= 0 - \lim_{x \rightarrow 0} \{x \cdot (-\ln x)^n\} + n \int_0^1 (-\ln x)^{n-1} dx = nJ(n-1)$$

$$(\because x = e^{-t}, \lim_{x \rightarrow 0} \{x \cdot (-\ln x)^n\} = \lim_{t \rightarrow \infty} \frac{t^n}{e^t} = 0 \text{ by l'Hospital's rule})$$

$$J_n = nJ_{n-1} = n(n-1)J_{n-2} = \dots = n(n-1) \dots 1 \cdot J_0 = n!$$

$$\therefore \int_0^1 (\ln x)^{2022} dx = J_{2022} = 2022! \blacksquare$$

30. $\int_1^e \frac{\ln x}{\sqrt{x}} dx = [\ln x \cdot 2\sqrt{x}]_1^e - \int_1^e \frac{2\sqrt{x}}{x} dx = 2\sqrt{e} - [4\sqrt{x}]_1^e = 4 - 2\sqrt{e} \blacksquare$

31. $I(\alpha) := \int_0^{\frac{\pi}{2}} \ln(a + \tan^2 x) dx$, $\frac{d}{d\alpha} I(\alpha) = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial \alpha} \ln(a + \tan^2 x) dx = \int_0^{\frac{\pi}{2}} \frac{1}{a + \tan^2 x} dx$

$$= \int_0^\infty \frac{1}{(t^2 + 1)(a + t^2)} dt = \frac{1}{a-1} \int_0^\infty \left(\frac{1}{t^2 + 1} - \frac{1}{t^2 + a} \right) dt \quad (t = \tan x, \quad dt = \sec^2 x dx)$$

$$= \frac{1}{a-1} \left[\tan^{-1} t - \frac{1}{\sqrt{a}} \tan^{-1} \left(\frac{t}{\sqrt{a}} \right) \right]_0^\infty = \frac{\pi}{2} \left(\frac{1}{a-1} - \frac{1}{\sqrt{a}(a-1)} \right)$$

$$I(\alpha) = \int \frac{\pi}{2} \left(\frac{1}{a-1} - \frac{1}{\sqrt{a}(a-1)} \right) da = \frac{\pi}{2} \int \left(\frac{1}{a-1} - \frac{1}{\sqrt{a}(a-1)} \right) dx$$

$$= \frac{\pi}{2} \left(\ln(a-1) + \ln \left(\frac{\sqrt{a}+1}{\sqrt{a}-1} \right) \right) = \pi \ln(\sqrt{a}+1)$$

$$(\because I(0) = 2 \int_0^{\frac{\pi}{2}} \ln(\tan x) dx = 2 \int_0^{\frac{\pi}{2}} \ln(\cot x) dx = -I(0) = 0)$$

$$\therefore \int_0^{\frac{\pi}{2}} \ln(2 + \tan^2 x) dx = \pi \ln(\sqrt{2}+1) \blacksquare$$

$$32. I(\alpha) := \int_0^\infty \frac{\alpha x - \sin(\alpha x)}{x^3(x^2+1)} dx, \quad \frac{d^3}{d\alpha^3} I(\alpha) = \int_0^\infty \frac{\partial^3}{\partial \alpha^3} \frac{\alpha x - \sin(\alpha x)}{x^3(x^2+1)} dx$$

$$= \int_0^\infty \frac{\partial^2}{\partial \alpha^2} \frac{1 - \cos(\alpha x)}{x^2(x^2+1)} dx = \int_0^\infty \frac{\partial}{\partial \alpha} \frac{\sin(\alpha x)}{x(x^2+1)} dx = \int_0^\infty \frac{\cos(\alpha x)}{x^2+1} dx$$

$$J(\alpha) := \int_{-\infty}^\infty \frac{\cos \alpha x}{x^2+1} dx = \left[\frac{\sin(\alpha x)}{\alpha(x^2+1)} \right]_{-\infty}^\infty + \frac{2}{\alpha} \int_{-\infty}^\infty \frac{x \sin(\alpha x)}{(x^2+1)^2} dx$$

$$= \frac{2}{\alpha} \int_{-\infty}^\infty \frac{x \sin(\alpha x)}{(x^2+1)^2} dx, \quad \alpha J(\alpha) = 2 \int_{-\infty}^\infty \frac{x \sin(\alpha x)}{(x^2+1)^2} dx$$

양변을 미분하면

$$\frac{d}{d\alpha}(\alpha J(\alpha)) = J(\alpha) + \alpha J'(\alpha) = 2 \int_{-\infty}^\infty \frac{\partial}{\partial \alpha} \frac{x \sin(\alpha x)}{(x^2+1)^2} dx = 2 \int_{-\infty}^\infty \frac{x^2 \cos(\alpha x)}{(x^2+1)^2} dx$$

$$= 2 \int_{-\infty}^\infty \frac{\cos(\alpha x)}{x^2+1} dx - 2 \int_{-\infty}^\infty \frac{\cos(\alpha x)}{(x^2+1)^2} dx = 2J(\alpha) - 2 \int_{-\infty}^\infty \frac{\cos(\alpha x)}{(x^2+1)^2} dx$$

$$\alpha J'(\alpha) - J(\alpha) = -2 \int_{-\infty}^\infty \frac{\cos(\alpha x)}{(x^2+1)^2} dx, \quad \text{다시 양변을 } \alpha \text{로 미분하면}$$

$$\alpha J''(\alpha) = -2 \int_{-\infty}^\infty \frac{\partial}{\partial \alpha} \frac{\cos(\alpha x)}{(x^2+1)^2} dx = 2 \int_{-\infty}^\infty \frac{x \sin(\alpha x)}{(x^2+1)^2} dx = \alpha J(\alpha)$$

$J''(\alpha) = J(\alpha)$ 라는 미분방정식을 얻는다.

$J(\alpha)$ 의 이계도함수가 $J(\alpha)$ 와 동일해야 하므로 $J(\alpha)$ 는 지수함수 또는 삼각함수들의 합으로 구성되어 있어야 하고, 상수 C_1, C_2, C_3, C_4 에 대하여

$J(\alpha) = C_1 e^{f(\alpha)} + C_2 e^{-f(\alpha)}$ 와 $J(\alpha) = C_3 \sin g(\alpha) + C_4 \cos g(\alpha)$ 의 형태를 모두 시험해보면 다음과 같은 해를 얻는다. (오일러공식 $e^{ix} = \cos x + i \sin x$ 를 적용하면 두 경우가 동일함을 알 수 있다.)

$$J(\alpha) = C_1 e^\alpha + C_2 e^{-\alpha} \quad (\alpha > 0)$$

$$\text{한편 } J(0) = C_1 + C_2 = \int_{-\infty}^{\infty} \frac{1}{x^2 + 1} dx = [\tan^{-1} x]_{-\infty}^{\infty} = \pi,$$

$$J(\alpha) = \frac{2}{\alpha} \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2 + 1)^2} dx \text{에서 } \lim_{\alpha \rightarrow \infty} J(\alpha) = 0, \quad C_1 = 0 \text{이고 } C_2 = \pi \text{이다.}$$

$$\therefore J(\alpha) = \frac{\pi}{e^\alpha}$$

$$\frac{d^3}{d\alpha^3} I(\alpha) = \int_0^\infty \frac{\cos(\alpha x)}{x^2 + 1} dx = \frac{1}{2} J(\alpha) = \frac{\pi}{2e^\alpha} \text{이고}$$

$$I(0) = I'(0) = I''(0) = 0 \text{이므로}$$

$$I''(\alpha) = \frac{\pi}{2}(1 - e^{-\alpha}), \quad I'(\alpha) = \frac{\pi}{2}(\alpha + e^{-\alpha} - 1), \quad I(\alpha) = \frac{\pi}{2} \left(\frac{1}{2} \alpha^2 - e^{-\alpha} - \alpha + 1 \right) \text{이다.}$$

한편 $x = 2t, dx = 2dt$ 의 치환을 하면

$$\begin{aligned} \int_0^\infty \frac{x - \sin x}{x^3(x^2 + 4)} dx &= \int_0^\infty \frac{2t - \sin(2t)}{8t^3(4t^2 + 4)} \cdot 2dt = \frac{1}{16} \int_0^\infty \frac{2t - \sin(2t)}{t^3(t^2 + 1)} dt = \frac{1}{16} I(2) \\ &= \frac{\pi}{32} (1 - e^{-2}) \blacksquare \end{aligned}$$

33. $u = \tan \frac{x}{2}, du = \frac{1}{2} \sec^2 \frac{x}{2} dx$

$$\int_0^\pi \frac{1 - \sin x}{1 + \sin x} dx = \int_0^\infty \frac{2\left(1 - \frac{2u}{u^2 + 1}\right)}{(u^2 + 1)\left(\frac{2u}{u^2 + 1} + 1\right)} du = \int_0^\infty \frac{2(u-1)^2}{(u^2 + 1)(u+1)^2} du$$

$$= 2 \int_0^\infty \left(\frac{2}{(u+1)^2} - \frac{1}{u^2 + 1}\right) du = 2 \left[-\frac{2}{u+1} - \tan^{-1} u\right]_0^\infty = 4 - \pi \blacksquare$$

$$34. \int_0^{\frac{3}{5}} \frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}} dx = \int_0^{\frac{3}{5}} \left(\frac{e^x}{(1-x)\sqrt{1-x^2}} + e^x \sqrt{\frac{1+x}{1-x}}\right) dx$$

$$= \int_0^{\frac{3}{5}} \left(e^x \left(\sqrt{\frac{1+x}{1-x}}\right)' + (e^x)' \sqrt{\frac{1+x}{1-x}}\right) dx = \left[e^x \sqrt{\frac{1+x}{1-x}}\right]_0^{\frac{3}{5}} = 2e^{\frac{3}{5}} - 1 \blacksquare$$

$$35. \int_0^{\frac{\pi}{2}} \sin 2022x \cdot \sin^{2020} x dx = \int_0^{\frac{\pi}{2}} \sin(2021x + x) \cdot \sin^{2020} x dx$$

$$= \int_0^{\frac{\pi}{2}} \sin 2021x \cdot \cos x \cdot \sin^{2020} x dx + \int_0^{\frac{\pi}{2}} \cos 2021x \cdot \sin^{2021} x dx$$

$$= \sin 2021x \cdot \frac{1}{2021} \sin^{2021} x - \frac{2021}{2021} \int_0^{\frac{\pi}{2}} \cos 2021x \cdot \sin^{2021} x dx + \int_0^{\frac{\pi}{2}} \cos 2021x \cdot \sin^{2021} x dx$$

$$= \left[\frac{1}{2021} \sin 2021x \cdot \sin^{2021} x\right]_0^{\frac{\pi}{2}} = \frac{1}{2021} \blacksquare$$

$$36. t = x^2 + \frac{1}{x^2} - 1, \quad dt = 2\left(x - \frac{1}{x^3}\right) dx$$

$$\int_1^{\sqrt{2}} \frac{x^4 - 1}{x^2 \sqrt{x^4 - x^2 + 1}} dx = \int_1^{\sqrt{2}} \frac{x^4 - 1}{x^3 \sqrt{x^2 + \frac{1}{x^2} - 1}} dx = \int_0^{\sqrt{2}} \frac{x - 1/x^3}{\sqrt{x^2 + \frac{1}{x^2} - 1}} dx$$

$$= \frac{1}{2} \int_1^{\frac{3}{2}} \frac{1}{\sqrt{t}} dt = \left[\sqrt{t}\right]_1^{\frac{3}{2}} = \frac{\sqrt{6}}{2} - 1 \blacksquare$$

$$37. u = \cos x, \quad du = -\sin x dx$$

$$\int_0^{\frac{\pi}{3}} \frac{\sin 2x}{2 + \cos x} dx = (-2) \int_0^{\frac{\pi}{3}} \frac{\cos x (-\sin x)}{2 + \cos x} dx = (-2) \int_1^{\frac{1}{2}} \frac{u}{2+u} du = 2 \int_{\frac{1}{2}}^1 \left(1 - \frac{2}{u+2}\right) du$$

$$= 2 \left[u - 2 \ln |u+2| \right]_{\frac{1}{2}}^1 = 2(1 - \ln 9) - 2 \left(\frac{1}{2} - 2 \ln \frac{5}{2} \right) = 1 - \ln \left(\frac{1296}{625} \right) \blacksquare$$

38. (19)의 reduction formula를 이용하면

$$\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx \quad \text{에서}$$

$$\int_0^{\frac{\pi}{4}} \tan^9 x dx = \left[\frac{\tan^8 x}{8} \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^7 x dx = \frac{1}{8} - \left[\frac{\tan^6 x}{6} \right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \tan^5 x dx$$

$$= \frac{1}{8} - \frac{1}{6} + \left[\frac{\tan^4 x}{4} \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^3 x dx = \frac{1}{8} - \frac{1}{6} + \frac{1}{4} - \left[\frac{\tan^2 x}{2} \right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \tan x dx$$

$$= \frac{1}{8} - \frac{1}{6} + \frac{1}{4} - \frac{1}{2} + [-\ln |\cos x|]_0^{\frac{\pi}{4}} = \frac{1}{8} - \frac{1}{6} + \frac{1}{4} - \frac{1}{2} + \ln \sqrt{2} = \frac{1}{2} \ln 2 - \frac{7}{24} \blacksquare$$

39. $I = \int_0^1 \frac{x-1}{(1+x^3)\ln x} dx = \frac{1}{2} \int_0^\infty \frac{x-1}{(1+x^3)\ln x} dx \quad (x \mapsto \frac{1}{x})$

$$J(a) := \int_0^\infty \frac{x^a - 1}{(1+x^3)\ln x} dx, \quad J'(a) = \frac{\partial}{\partial a} J(a) = \int_0^\infty \frac{x^a}{1+x^3} dx = \frac{\pi}{3} \operatorname{csc} \left(\frac{\pi(a+1)}{3} \right)$$

$$\therefore I = \frac{1}{2} J(1) = \frac{1}{2} \int_0^1 J'(a) da = \frac{\pi}{6} \int_0^1 \operatorname{csc} \left(\frac{\pi(a+1)}{3} \right) da = \frac{\ln 3}{2} \blacksquare$$

40. sol 1) $u = \frac{\sin t}{\sqrt{\cos t}}, \quad du = \frac{1 + \cos^2 t}{2 \cos^{3/2} t} dt$

$$\int_0^{\frac{\pi}{2}} \frac{(1 + \sec^2 t) \sqrt{\sec t}}{(1 + \sec t)^2 - 2} dt = \int_0^{\frac{\pi}{2}} \frac{1 + \cos^2 t}{\sqrt{\cos t} \{(\cos t + 1)^2 - 2 \cos^2 t\}} dt$$

$$= \int_0^{\frac{\pi}{2}} \frac{1 + \cos^2 t}{\sqrt{\cos t} (1 + 2 \cos t - \cos^2 t)} dt = 2 \int_0^{\frac{\pi}{2}} \frac{\cos t}{1 + 2 \cos t - \cos^2 t} \cdot \frac{1 + \cos^2 t}{2 \cos^{3/2} t} dt$$

$$\begin{aligned}
&= 2 \int_0^{\frac{\pi}{2}} \frac{1}{\frac{1}{\cos t} + 2 - \cos t} \cdot \frac{1 + \cos^2 t}{2 \cos^{3/2} t} dt = 2 \int_0^{\frac{\pi}{2}} \frac{1}{\frac{1 - \cos^2 t}{\cos t} + 2} \cdot \frac{1 + \cos^2 t}{2 \cos^{3/2} t} dt \\
&= 2 \int_0^{\frac{\pi}{2}} \frac{1}{\frac{\sin^2 t}{\cos t} + 2} \cdot \frac{1 + \cos^2 t}{2 \cos^{3/2} t} dt = 2 \int_0^{\infty} \frac{1}{u^2 + 2} du = 2 \left[\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x}{\sqrt{2}} \right) \right]_0^{\infty} = \frac{\pi}{\sqrt{2}} \blacksquare
\end{aligned}$$

sol 2) $\text{sect} = x^2$, $\text{sectant} dt = 2x dx$, $u = \sqrt{x^2 - \frac{1}{x^2}}$, $du = \frac{x^4 + 1}{x^2 \sqrt{x^4 - 1}}$

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \frac{(1 + \sec^2 t) \sqrt{\text{sect}}}{(1 + \text{sect})^2 - 2} dt = 2 \int_1^{\infty} \frac{x^4 + 1}{\sqrt{x^4 - 1} (x^4 + 2x^2 - 1)} dx = 2 \int_0^{\infty} \frac{1}{u^2 + 2} du \\
&= 2 \left[\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x}{\sqrt{2}} \right) \right]_0^{\infty} = \frac{\pi}{\sqrt{2}} \blacksquare
\end{aligned}$$

$$\begin{aligned}
41. \int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - \sin 2x} dx &= \int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - 2 \sin x \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \cos^2 x - 2 \sin x \cos x + \sin^2 x} dx \\
&= \int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + (\cos x - \sin x)^2} dx = \frac{1}{2} \left[\int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + (\cos x - \sin x)^2} dx + \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + (\cos x - \sin x)^2} dx \right] \\
(\because x \mapsto \frac{\pi}{2} - x : \int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + (\cos x - \sin x)^2} dx &= \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + (\cos x - \sin x)^2} dx) \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{1 + (\cos x - \sin x)^2} dx = -\frac{1}{2} [\tan^{-1}(\cos x - \sin x)]_0^{\frac{\pi}{2}} = \frac{\pi}{4} \blacksquare
\end{aligned}$$

$$\begin{aligned}
42. \int_0^{\infty} (\ln(e^x + 1) - x) dx &= \int_0^{\infty} \ln \left(\frac{e^x + 1}{e^x} \right) dx = \int_0^{\infty} \ln(1 + e^{-x}) dx \\
&= \int_0^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} e^{-nx}}{n} dx = \sum_{n=1}^{\infty} \int_0^{\infty} \frac{(-1)^{n-1} e^{-nx}}{n} dx = \sum_{n=1}^{\infty} \left[\frac{(-1)^n e^{-nx}}{n^2} \right]_0^{\infty} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} - 2 \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\zeta(2)}{2} = \frac{\pi^2}{12} \blacksquare
\end{aligned}$$

$$\begin{aligned}
43. \quad I &= \int_0^\infty \frac{x}{e^x + 1} dx = \int_1^\infty \frac{\ln t}{t(1+t)} dt \quad (x = \ln t, \quad dx = \frac{1}{t} dt) \\
&= - \int_0^1 \frac{\ln u}{1+u} du \quad (u = \frac{1}{t}, \quad du = -\frac{1}{u^2} du) \\
&= \int_0^1 \frac{\ln u}{1-u} du - \int_0^1 \frac{2u \ln u}{1-u^2} du = \int_0^1 \frac{\ln u}{1-u} du - \frac{1}{2} \int_0^1 \frac{\ln y}{1-y} dy \quad (y = u^2, \quad dy = 2u du) \\
&= \frac{1}{2} \int_0^1 \frac{\ln u}{1-u} du
\end{aligned}$$

$(1+x)^{-1}$ 의 이항급수를 구하고 양변을 적분하면

$$(1+x)^{-1} = \sum_{n=0}^{\infty} \binom{-1}{n} \cdot x^n = \sum_{n=0}^{\infty} \prod_{i=0}^{n-1} (-1-i) \cdot \frac{x^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \cdot x^n \quad (|x| < 1)$$

$$\int (1+x)^{-1} dx = \ln |1+x| = \int \sum_{n=0}^{\infty} (-1)^n x^n dx + C = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + C$$

$x=0$ 을 대입하면 $0 = 0 + C$ 에서 $C=0$ 이다.

$$\int_0^1 \frac{\ln x}{x-1} dx = \int_{-1}^0 \frac{\ln(t+1)}{t} dt = \int_{-1}^0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} t^n dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} t^{n+1} \right]_{-1}^0$$

$$= \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} 0^{n+1} \right] - \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} (-1)^{n+1} \right] = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$$

$$= \zeta(2) = \frac{\pi^2}{6} \quad (\because \text{바젤 문제에 의해 } \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6})$$

$$I = \int_0^\infty \frac{x}{e^x + 1} dx = \int_1^\infty \frac{\ln t}{t(1+t)} dt = \frac{1}{2} \int_0^1 \frac{\ln u}{1-u} du = \frac{\pi^2}{12} \quad \blacksquare$$

$$44. \quad \int_0^{\frac{\pi}{2}} \frac{\ln(\tan x)}{1 - \tan x + \tan^2 x} dx = \int_0^\infty \frac{\ln t}{(1-t+t^2)(1+t^2)} dt \quad (t = \tan x, \quad dt = \sec^2 x dx)$$

$$= \int_0^1 \frac{\ln t}{(1-t+t^2)(1+t^2)} dt + \int_1^\infty \frac{\ln t}{(1-t+t^2)(1+t^2)} dt$$

$$\begin{aligned}
&= \int_0^1 \frac{\ln t}{(1-t+t^2)(1+t^2)} dt - \int_0^1 \frac{u^2 \ln u}{(1-u+u^2)(1+u^2)} du \quad (u = \frac{1}{t}, \quad du = -\frac{1}{t^2} dt) \\
&= \int_0^1 \frac{(1-u^2) \ln u}{(1-u+u^2)(1+u^2)} du = \int_0^1 \frac{2u \ln u}{1+u^2} du - \int_0^1 \frac{(2u-1) \ln u}{1-u+u^2} du \\
&= [\ln u \ln(u^2+1)]_0^1 - \int_0^1 \frac{1}{u} \ln(u^2+1) du - [\ln u \ln(1-u+u^2)]_0^1 + \int_0^1 \frac{1}{u} \ln(1-u+u^2) du \\
&= -\int_0^1 \frac{\ln(1+u^2)}{u} du + \int_0^1 \frac{\ln(1-u+u^2)}{u} du \\
&= -\int_0^1 \frac{\ln(1+u^2)}{u^2} u du + \int_0^1 \frac{\ln(1+u^3)}{u} du - \int_0^1 \frac{\ln(1+u)}{u} du \\
&= -\frac{1}{2} \int_0^1 \frac{\ln(1+y)}{y} dy + \int_0^1 \frac{\ln(1+u^3)}{u^3} u^2 du - \int_0^1 \frac{\ln(1+u)}{u} du \quad (y = u^2, \quad dy = 2u du) \\
&= -\frac{3}{2} \int_0^1 \frac{\ln(1+u)}{u} du + \frac{1}{3} \int_0^1 \frac{\ln(1+z)}{z} dz \quad (z = u^3, \quad dz = 3u^2 du) \\
&= -\frac{7}{6} \int_0^1 \frac{\ln(1+u)}{u} du = -\frac{7}{6} \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n u^n}{n+1} du = -\frac{7}{6} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \int_0^1 u^n du \\
&= -\frac{7}{6} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} = -\frac{7}{6} \left(\sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \right) = -\frac{7}{12} \sum_{n=1}^{\infty} \frac{1}{n^2} \\
&= -\frac{7}{12} \zeta(2) = -\frac{7\pi^2}{72} \quad (\because \text{바젤 문제에 의해 } \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6})
\end{aligned}$$

45. $t = x - \frac{1}{x}, \quad dt = \left(1 + \frac{1}{x^2}\right) dx, \quad \frac{dx}{1+x^2} = \frac{x^2}{(1+x^2)^2} dt = \frac{dt}{(x+1/x)^2} = \frac{dt}{t^2+4}$

$$I = \int_{-\infty}^{\infty} \frac{\cos\left(x - \frac{1}{x}\right)}{1+x^2} dx = 2 \int_0^{\infty} \frac{\cos\left(x - \frac{1}{x}\right)}{1+x^2} dx = 2 \int_{-\infty}^{\infty} \frac{\cos t}{t^2+4} dt$$

$$J(\alpha) := \int_{-\infty}^{\infty} \frac{\cos \alpha x}{x^2+1} dx = \left[\frac{\sin(\alpha x)}{\alpha(x^2+1)} \right]_{-\infty}^{\infty} + \frac{2}{\alpha} \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2+1)^2} dx$$

$$= \frac{2}{\alpha} \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2+1)^2} dx, \quad \alpha J(\alpha) = 2 \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2+1)^2} dx$$

양변을 미분하면

$$\frac{d}{d\alpha}(\alpha J(\alpha)) = J(\alpha) + \alpha J'(\alpha) = 2 \int_{-\infty}^{\infty} \frac{\partial}{\partial \alpha} \frac{x \sin(\alpha x)}{(x^2+1)^2} dx = 2 \int_{-\infty}^{\infty} \frac{x^2 \cos(\alpha x)}{(x^2+1)^2} dx$$

$$= 2 \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{x^2+1} dx - 2 \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{(x^2+1)^2} dx = 2J(\alpha) - 2 \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{(x^2+1)^2} dx$$

$$\alpha J'(\alpha) - J(\alpha) = -2 \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{(x^2+1)^2} dx, \text{ 다시 양변을 } \alpha \text{로 미분하면}$$

$$\alpha J''(\alpha) = -2 \int_{-\infty}^{\infty} \frac{\partial}{\partial \alpha} \frac{\cos(\alpha x)}{(x^2+1)^2} dx = 2 \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2+1)^2} dx = \alpha J(\alpha)$$

$J''(\alpha) = J(\alpha)$ 라는 미분방정식을 얻는다.

$J(\alpha)$ 의 이계도함수가 $J(\alpha)$ 와 동일해야 하므로 $J(\alpha)$ 는 지수함수 또는 삼각함수들의 합으로 구성되어 있어야 하고, 상수 C_1, C_2, C_3, C_4 에 대하여

$J(\alpha) = C_1 e^{f(\alpha)} + C_2 e^{-f(\alpha)}$ 와 $J(\alpha) = C_3 \sin g(\alpha) + C_4 \cos g(\alpha)$ 의 형태를 모두 시험해보면 다음과 같은 해를 얻는다. (오일러공식 $e^{ix} = \cos x + i \sin x$ 를 적용하면 두 경우가 동일함을 알 수 있다.)

$$J(\alpha) = C_1 e^{\alpha} + C_2 e^{-\alpha} \quad (\alpha \geq 0)$$

$$\text{한편 } J(0) = C_1 + C_2 = \int_{-\infty}^{\infty} \frac{1}{x^2+1} dx = [\tan^{-1} x]_{-\infty}^{\infty} = \pi,$$

$$J(\alpha) = \frac{2}{\alpha} \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2+1)^2} dx \text{에서 } \lim_{\alpha \rightarrow \infty} J(\alpha) = 0, \quad C_1 = 0 \text{이고 } C_2 = \pi \text{이다.}$$

$$J(\alpha) = \int_{-\infty}^{\infty} \frac{\cos \alpha x}{x^2+1} dx = \frac{\pi}{e^{\alpha}}$$

$$\therefore I = 2 \int_{-\infty}^{\infty} \frac{\cos t}{t^2+4} dt = 4 \int_{-\infty}^{\infty} \frac{\cos 2u}{4u^2+4} du \quad (t = 2u, \quad dt = 2du)$$

$$= J(2) = \frac{\pi}{e^2} \blacksquare$$

$$46. I(\alpha) := \int_{-\infty}^{\infty} \frac{\cos \alpha x}{x^2 + 1} dx = \left[\frac{\sin(\alpha x)}{\alpha(x^2 + 1)} \right]_{-\infty}^{\infty} + \frac{2}{\alpha} \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2 + 1)^2} dx$$

$$= \frac{2}{\alpha} \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2 + 1)^2} dx, \quad \alpha I(\alpha) = 2 \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2 + 1)^2} dx$$

양변을 미분하면

$$\frac{d}{d\alpha}(\alpha I(\alpha)) = I(\alpha) + \alpha I'(\alpha) = 2 \int_{-\infty}^{\infty} \frac{\partial}{\partial \alpha} \frac{x \sin(\alpha x)}{(x^2 + 1)^2} dx = 2 \int_{-\infty}^{\infty} \frac{x^2 \cos(\alpha x)}{(x^2 + 1)^2} dx$$

$$= 2 \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{x^2 + 1} dx - 2 \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{(x^2 + 1)^2} dx = 2I(\alpha) - 2 \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{(x^2 + 1)^2} dx$$

$$\alpha I'(\alpha) - I(\alpha) = -2 \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{(x^2 + 1)^2} dx, \text{ 다시 양변을 } \alpha \text{로 미분하면}$$

$$\alpha I''(\alpha) = -2 \int_{-\infty}^{\infty} \frac{\partial}{\partial \alpha} \frac{\cos(\alpha x)}{(x^2 + 1)^2} dx = 2 \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2 + 1)^2} dx = \alpha I(\alpha)$$

$I''(\alpha) = I(\alpha)$ 라는 미분방정식을 얻는다.

$I(\alpha)$ 의 이계도함수가 $I(\alpha)$ 와 동일해야 하므로 $I(\alpha)$ 는 지수함수 또는 삼각함수들의 합으로 구성되어 있어야 하고, 상수 C_1, C_2, C_3, C_4 에 대하여

$I(\alpha) = C_1 e^{f(\alpha)} + C_2 e^{-f(\alpha)}$ 와 $I(\alpha) = C_3 \sin g(\alpha) + C_4 \cos g(\alpha)$ 의 형태를 모두 시험해보면 다음과 같은 해를 얻는다. (오일러공식 $e^{ix} = \cos x + i \sin x$ 를 적용하면 두 경우가 동일함을 알 수 있다.)

$$I(\alpha) = C_1 e^{\alpha} + C_2 e^{-\alpha} \quad (\alpha \geq 0)$$

$$\text{한편 } I(0) = C_1 + C_2 = \int_{-\infty}^{\infty} \frac{1}{x^2 + 1} dx = [\tan^{-1} x]_{-\infty}^{\infty} = \pi,$$

$$I(\alpha) = \frac{2}{\alpha} \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2 + 1)^2} dx \text{에서 } \lim_{\alpha \rightarrow \infty} I(\alpha) = 0, \quad C_1 = 0 \text{이고 } C_2 = \pi \text{이다.}$$

$$I(\alpha) = \frac{\pi}{e^\alpha}, \quad \therefore \int_0^\infty \frac{\cos x}{x^2+1} dx = \frac{1}{2}I(1) = \frac{\pi}{2e} \blacksquare$$

$$\begin{aligned} 47. \quad I(\alpha) &:= \int_0^\infty \frac{x \sin(\alpha x)}{x^2+1} dx = \int_0^\infty \frac{x^2 \sin(\alpha x)}{x(x^2+1)} dx \quad (\alpha > 0) \\ &= \int_0^\infty \frac{(1+x^2-1)\sin(\alpha x)}{x(x^2+1)} dx = \int_0^\infty \frac{\sin(\alpha x)}{x} dx - \int_0^\infty \frac{\sin(\alpha x)}{x(1+x^2)} dx \\ &= \int_0^\infty \frac{\sin t}{t} dt - \int_0^\infty \frac{\sin(\alpha x)}{x(1+x^2)} dx \quad (t = \alpha x, \quad dt = \alpha dx) \\ &= \frac{\pi}{2} - \int_0^\infty \frac{\sin(\alpha x)}{x(1+x^2)} dx \quad (\ast) \end{aligned}$$

$$\frac{d}{d\alpha} I(\alpha) = - \int_0^\infty \frac{\partial}{\partial \alpha} \frac{\sin(\alpha x)}{x(1+x^2)} dx = - \int_0^\infty \frac{\cos(\alpha x)}{1+x^2} dx$$

$$\frac{d^2}{d\alpha^2} I(\alpha) = - \int_0^\infty \frac{\partial}{\partial \alpha} \frac{\cos(\alpha x)}{1+x^2} dx = \int_0^\infty \frac{x \sin(\alpha x)}{1+x^2} dx = I(\alpha)$$

$I''(\alpha) = I(\alpha)$ 라는 미분방정식을 얻는다.

$I(\alpha)$ 의 이계도함수가 $I(\alpha)$ 와 동일해야 하므로 $I(\alpha)$ 는 지수함수 또는 삼각함수들의 합으로 구성되어 있어야 하고, 상수 C_1, C_2, C_3, C_4 에 대하여

$I(\alpha) = C_1 e^{f(\alpha)} + C_2 e^{-f(\alpha)}$ 와 $I(\alpha) = C_3 \sin g(\alpha) + C_4 \cos g(\alpha)$ 의 형태를 모두 시험해보면 다음과 같은 해를 얻는다. (오일러공식 $e^{ix} = \cos x + i \sin x$ 를 적용하면 두 경우가 동일함을 알 수 있다.)

$$I(\alpha) = C_1 e^\alpha + C_2 e^{-\alpha} \quad (\alpha > 0)$$

한편

$$\lim_{\alpha \rightarrow 0^+} I(\alpha) = \lim_{\alpha \rightarrow 0^+} \left(\frac{\pi}{2} - \int_0^\infty \frac{\sin(\alpha x)}{x(1+x^2)} dx \right) = \frac{\pi}{2} \text{에서 } C_1 + C_2 = \frac{\pi}{2},$$

$$\lim_{\alpha \rightarrow 0^+} I'(\alpha) = \lim_{\alpha \rightarrow 0^+} \left(- \int_0^\infty \frac{\cos(\alpha x)}{1+x^2} dx \right) = -\frac{\pi}{2} \text{에서 } C_1 - C_2 = -\frac{\pi}{2} \text{를 얻는다.}$$

따라서 $C_1 = 0$, $C_2 = \frac{\pi}{2}$ 이고

$$I(\alpha) = \int_0^{\infty} \frac{x \sin(\alpha x)}{x^2 + 1} dx = \frac{\pi}{2e^\alpha},$$

$$\therefore \int_0^{\infty} \frac{x \sin x}{x^2 + 1} dx = I(1) = \frac{\pi}{2e} \blacksquare$$

$$\ast J(\alpha) := \int_0^{\infty} \frac{\sin x}{x} e^{-\alpha x} dx, \quad \frac{d}{d\alpha} J(\alpha) = \int_0^{\infty} \frac{\partial}{\partial \alpha} \frac{\sin x}{x} e^{-\alpha x} dx = - \int_0^{\infty} e^{-\alpha x} \sin x dx$$

$$= - [-e^{-\alpha x} \cos x]_0^{\infty} + \int_0^{\infty} \alpha e^{-\alpha x} \cos x dx = -1 + [\alpha e^{-\alpha x} \sin x]_0^{\infty} + \int_0^{\infty} \alpha^2 e^{-\alpha x} \sin x dx$$

$$= -1 - \alpha^2 J'(\alpha), \quad J'(\alpha) = - \frac{1}{1 + \alpha^2}$$

$$J(\alpha) = \int J'(\alpha) d\alpha = C - \tan^{-1} \alpha \quad (C \in \mathbb{R}), \quad 0 = \lim_{\alpha \rightarrow \infty} J(\alpha) = C - \frac{\pi}{2} \text{에서 } C = \frac{\pi}{2} \text{이다.}$$

$$\therefore \int_0^{\infty} \frac{\sin x}{x} dx = J(0) = \frac{\pi}{2} \blacksquare$$

$$48. \int_0^{\infty} \frac{\ln(1+x)}{x(x^2+1)} dx = \int_0^1 \frac{\ln(1+x)}{x(x^2+1)} dx + \int_1^{\infty} \frac{\ln(1+x)}{x(x^2+1)} dx$$

$$= \int_0^1 \frac{\ln(1+x)}{x(x^2+1)} dx + \int_1^0 \frac{t \ln(1+t^{-1})}{t^{-2}+1} \left(-\frac{1}{t^2} dt\right) \left(t = \frac{1}{x}, dt = -\frac{1}{x^2} dx\right)$$

$$= \int_0^1 \frac{\ln(1+x)}{x(x^2+1)} dx + \int_0^1 \frac{t \ln(1+t^{-1})}{1+t^2} dt$$

$$= \int_0^1 \frac{\ln(1+x)}{x(x^2+1)} dx + \int_0^1 \frac{t \ln(1+t)}{1+t^2} dt - \int_0^1 \frac{t \ln t}{1+t^2} dt$$

$$= \int_0^1 \left(t + \frac{1}{t}\right) \frac{\ln(1+t)}{1+t^2} dt - \int_0^1 \frac{t \ln t}{1+t^2} dt$$

$$= \int_0^1 \frac{\ln(1+t)}{t} dt - \frac{1}{4} \int_0^1 \frac{\ln u}{1+u} du \quad (u = t^2, du = 2t dt)$$

$$\begin{aligned}
&= \int_0^1 \frac{\ln(1+t)}{t} dt - \frac{1}{4} \left([\ln t \ln(1+t)]_0^1 - \int_0^1 \frac{\ln(1+t)}{t} dt \right) = \frac{5}{4} \int_0^1 \frac{\ln(1+t)}{t} dt \\
&= \frac{5}{4} \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n u^n}{n+1} du = \frac{5}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \int_0^1 u^n du = \frac{5}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} \\
&= \frac{5}{4} \left(\sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \right) = \frac{5}{8} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{5}{8} \zeta(2) = \frac{5\pi^2}{48} \blacksquare
\end{aligned}$$

$$(\because \text{바젤 문제에 의해 } \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6})$$

※ $x = \tan t$ 의 치환을 한 후 $I(\alpha) := \int_0^{\frac{\pi}{2}} \frac{\ln(1+\alpha \tan t)}{\tan t} dt$ 로 정의하면 파인만 테크닉을 적용하여

$$I(\alpha) = \frac{\pi + 2\alpha \ln \alpha}{2(1+\alpha^2)}, \quad I(1) = \int_0^{\infty} \frac{\ln(1+x)}{x(1+x^2)} dx = \frac{5\pi^2}{48}$$

을 얻을 수 있다.

49. $v = \tan u, dv = \sec^2 u du$

$$\begin{aligned}
I(\alpha) &:= \int_0^{\frac{\pi}{2}} \frac{\sin(u + \alpha \tan u)}{\sin u} du = \int_0^{\frac{\pi}{2}} \left(\cos(\alpha \tan u) + \frac{\sin(\alpha \tan u)}{\tan u} \right) du \\
&= \int_0^{\infty} \frac{\cos(\alpha v) + \frac{\sin(\alpha v)}{v}}{1+v^2} dv, \\
\frac{d}{d\alpha} I(\alpha) &= \int_0^{\infty} \frac{\partial}{\partial \alpha} \frac{\cos(\alpha v) + \frac{\sin(\alpha v)}{v}}{1+v^2} dv = \int_0^{\infty} \frac{\cos(\alpha v) - v \sin(\alpha v)}{1+v^2} dv \\
&= \int_0^{\infty} \frac{\cos(\alpha v)}{1+v^2} dv - \int_0^{\infty} \frac{v \sin(\alpha v)}{1+v^2} dv = \frac{\pi}{2e} - \frac{\pi}{2e} = 0 \quad (\because \#46, \#47) \\
\therefore I(\alpha) &= C = I(0) = \frac{\pi}{2}
\end{aligned}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin(u + \sqrt{\pi} \tan u)}{\sin u} du = I(\sqrt{\pi}) = \frac{\pi}{2} \blacksquare$$

$$50. I(\alpha) := \int_0^\alpha \sin^2(x - \sqrt{\alpha^2 - x^2}) dx = \int_{\frac{\pi}{2}}^0 \sin^2(\alpha \cos t - \alpha \sin t) \cdot (-\alpha \sin t dt)$$

$$= \int_0^{\frac{\pi}{2}} \sin^2(\alpha \cos t - \alpha \sin t) \alpha \sin t dt \quad (x = \alpha \cos t, \quad dx = -\alpha \sin t dt)$$

$$I(\alpha) := \int_0^\alpha \sin^2(x - \sqrt{\alpha^2 - x^2}) dx = \int_0^{\frac{\pi}{2}} \sin^2(\alpha \sin t - \alpha \cos t) \alpha \cos t dt$$

$$= \int_0^{\frac{\pi}{2}} \sin^2(\alpha \cos t - \alpha \sin t) \alpha \cos t dt \quad (x = \alpha \sin t, \quad dx = \alpha \cos t dt)$$

$$I(\alpha) = \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin^2(\alpha(\sin t - \cos t)) \alpha(\sin t + \cos t) dt$$

$$= \frac{1}{2} \int_{-\alpha}^\alpha \sin^2 u du \quad (u = \alpha(\sin t - \cos t), \quad du = \alpha(\cos t + \sin t) dt)$$

$$= \frac{1}{4} \int_{-\alpha}^\alpha (1 - \cos 2u) du = \frac{1}{4} \left[u - \frac{1}{2} \sin 2u \right]_{-\alpha}^\alpha = \frac{\alpha}{2} - \frac{1}{4} \sin 2\alpha$$

$$\therefore \int_0^\pi \sin^2(x - \sqrt{\pi^2 - x^2}) dx = I(\pi) = \frac{\pi}{2} \blacksquare$$

$$51. I(\alpha) := \int_0^\infty e^{-x^2} \cos(\alpha x) dx$$

$$\frac{d}{d\alpha} I(\alpha) = \int_0^\infty \frac{\partial}{\partial \alpha} e^{-x^2} \cos(\alpha x) dx = - \int_0^\infty x e^{-x^2} \sin(\alpha x) dx$$

$$= - \left(\left[-\frac{1}{2} e^{-x^2} \sin(\alpha x) \right]_0^\infty + \frac{\alpha}{2} \int_0^\infty e^{-x^2} \cos(\alpha x) dx \right) = -\frac{\alpha}{2} I(\alpha)$$

$I'(\alpha) = -\frac{\alpha}{2} I(\alpha)$ 의 미분방정식을 얻는다. 이를 풀기 위해 변수를 분리하고 양변을 적분하면

$$\frac{1}{I(\alpha)}dI(\alpha) = -\frac{\alpha}{2}d\alpha, \quad \int \frac{1}{I(\alpha)}dI(\alpha) = -\int \frac{\alpha}{2}d\alpha, \quad \ln|I(\alpha)| + C = -\frac{1}{4}\alpha^2$$

$$I(0) = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \quad (\because \text{가우스 적분}) \quad \text{이므로 } C = -\ln\left(\frac{\sqrt{\pi}}{2}\right)$$

$$I(\alpha) = \int_0^\infty e^{-x^2} \cos(\alpha x) dx = \frac{\sqrt{\pi}}{2} e^{-\frac{\alpha^2}{4}}$$

$$\therefore \int_0^\infty e^{-x^2} \cos 2x dx = I(2) = \frac{\sqrt{\pi}}{2e} \quad \blacksquare$$

$$52. I(\alpha) := \int_0^{\frac{\pi}{2}} \cos(\alpha \tan x) dx, \quad t = \tan x, \quad dt = \sec^2 x dx, \quad dx = \frac{dt}{1+t^2}$$

$$I(\alpha) = \int_0^{\frac{\pi}{2}} \cos(\alpha \tan x) dx = \int_0^\infty \frac{\cos \alpha t}{1+t^2} dt = \frac{1}{\alpha} \left[\frac{\sin \alpha t}{1+t^2} \right]_0^\infty + \frac{1}{\alpha} \int_0^\infty \frac{2t \sin \alpha t}{(1+t^2)^2} dt$$

$$= \frac{1}{\alpha} \int_0^\infty \frac{2t \sin \alpha t}{(1+t^2)^2} dt, \quad \alpha I(\alpha) \text{에 대하여 파인만 적분 테크닉을 이용하면}$$

$$\frac{d}{d\alpha}(\alpha I(\alpha)) = I(\alpha) + \alpha I'(\alpha) = \int_0^\infty \frac{\partial}{\partial \alpha} \left(\frac{2t \sin \alpha t}{(1+t^2)^2} \right) dt = \int_0^\infty \frac{2t^2 \cos \alpha t}{(1+t^2)^2} dt$$

$$= \int_0^\infty \frac{2(t^2+1-1)\cos \alpha t}{(1+t^2)^2} dt = 2 \int_0^\infty \frac{\cos \alpha t}{1+t^2} dt - \int_0^\infty \frac{2\cos \alpha t}{(1+t^2)^2} dt$$

$$= 2I(\alpha) - \int_0^\infty \frac{2\cos \alpha t}{(1+t^2)^2} dt.$$

다시 양변을 α 에 대하여 미분하면

$$I'(\alpha) + \alpha I''(\alpha) + I'(\alpha) = 2I'(\alpha) - \int_0^\infty \frac{\partial}{\partial \alpha} \frac{2\cos \alpha t}{(1+t^2)^2} dt = 2I'(\alpha) + \int_0^\infty \frac{2t \sin \alpha t}{(1+t^2)^2} dt$$

$$\alpha I''(\alpha) = \int_0^\infty \frac{2t \sin \alpha t}{(1+t^2)^2} dt = \alpha I(\alpha), \quad I''(\alpha) = I(\alpha) \text{라는 미분방정식을 얻는다.}$$

$I(\alpha)$ 의 이계도함수가 $I(\alpha)$ 와 동일해야 하므로 $I(\alpha)$ 는 지수함수 또는 삼각함수들의 합으로 구성되어 있어야 하고, 상수 C_1, C_2, C_3, C_4 에 대하여

$I(\alpha) = C_1 e^{f(\alpha)} + C_2 e^{-f(\alpha)}$ 와 $I(\alpha) = C_3 \sin g(\alpha) + C_4 \cos g(\alpha)$ 의 형태를 모두 시험해보면 다음과 같은 해를 얻는다. (오일러공식 $e^{ix} = \cos x + i \sin x$ 를 적용하면 두 경우가 동일함을 알 수 있다.)

$$I(\alpha) = C_1 e^\alpha + C_2 e^{-\alpha} \quad (\alpha \geq 0)$$

이때 $I(\alpha)$ 는 $\cos(\alpha \tan x)$ 를 0부터 $\frac{\pi}{2}$ 까지 정적분한 값이므로 모든 $\alpha \geq 0$ 에 대하여 유계(bounded)이고, 따라서 $C_1 = 0$ 이다. 또한 $\alpha = 0$ 을 대입하면 $I(0) = \frac{\pi}{2} = C_2$ 를 얻는다.

$$I(\alpha) = \int_0^{\frac{\pi}{2}} \cos(\alpha \tan x) dx = \frac{\pi}{2e^\alpha},$$

$$\therefore \int_0^{\frac{\pi}{2}} \cos(\tan x) dx = I(1) = \frac{\pi}{2e} \blacksquare$$

53. $I(\alpha) := \int_0^{\frac{\pi}{2}} \frac{\tan^{-1}(\alpha \tan x)}{\tan x} dx$

$$\frac{d}{d\alpha} I(\alpha) = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial \alpha} \frac{\tan^{-1}(\alpha \tan x)}{\tan x} dx = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \alpha^2 \tan^2 x} dx$$

$$= \int_0^\infty \frac{1}{(1 + \alpha^2 t^2)(1 + t^2)} dt \quad (t = \tan x, \quad dt = \sec^2 x dx)$$

$$= \frac{1}{\alpha^2 - 1} \int_0^\infty \left(\frac{\alpha^2}{1 + \alpha^2 t^2} - \frac{1}{1 + t^2} \right) dt = \frac{1}{\alpha^2 - 1} [\alpha \tan^{-1}(\alpha t) - \tan^{-1} t]_0^\infty = \frac{\pi}{2(\alpha + 1)}$$

$$I(\alpha) = \frac{\pi}{2} \ln |\alpha + 1| + C, \quad C = I(0) = 0 \text{에서 } I(\alpha) = \frac{\pi}{2} \ln |\alpha + 1|$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{x}{\tan x} dx = I(1) = \frac{\pi}{2} \ln 2 \blacksquare$$

54. $I(\alpha) := \int_0^\pi \ln(1 - 2\alpha \cos x + \alpha^2) dx, \quad \frac{d}{d\alpha} I(\alpha) = \int_0^\pi \frac{\partial}{\partial \alpha} \ln(1 - 2\alpha \cos x + \alpha^2) dx$

$$= \int_0^\pi \frac{2(\alpha - \cos x)}{1 - 2\alpha \cos x + \alpha^2} dx = \frac{1}{\alpha} \int_0^\pi \frac{2(\alpha^2 - \alpha \cos x)}{1 - 2\alpha \cos x + \alpha^2} dx$$

$$\begin{aligned}
&= \frac{1}{\alpha} \int_0^\pi \left(\frac{2(\alpha^2 - \alpha \cos x)}{1 - 2\alpha \cos x + \alpha^2} - 1 + 1 \right) dx \\
&= \frac{1}{\alpha} \int_0^\pi \left(\frac{2(\alpha^2 - \alpha \cos x)}{1 - 2\alpha \cos x + \alpha^2} - \frac{1 - 2\alpha \cos x + \alpha^2}{1 - 2\alpha \cos x + \alpha^2} + 1 \right) dx \\
&= \frac{1}{\alpha} \int_0^\pi \left(1 - \frac{1 - \alpha^2}{1 + \alpha^2 - 2\alpha \cos x} \right) dx = \frac{\pi}{\alpha} - \frac{1 - \alpha^2}{\alpha(1 + \alpha^2)} \int_0^\pi \left(\frac{1}{1 - \frac{2\alpha}{1 + \alpha^2} \cos x} \right) dx \\
&= \frac{\pi}{\alpha} - \frac{1 - \alpha^2}{\alpha(1 + \alpha^2)} \int_0^\infty \frac{2}{(1 + t^2) \left(1 - \frac{2\alpha}{1 + \alpha^2} \frac{1 - t^2}{1 + t^2} \right)} dt \quad (t = \tan \frac{x}{2}, \quad dt = \frac{1}{2} \sec^2 \frac{x}{2} dx) \\
&= \frac{\pi}{\alpha} - \frac{1 - \alpha^2}{\alpha(1 + \alpha^2)} \int_0^\infty \frac{2}{\left(1 - \frac{2\alpha}{1 + \alpha^2} \right) + \left(1 + \frac{2\alpha}{1 + \alpha^2} \right) t^2} dt \\
&= \frac{\pi}{\alpha} - \frac{1 - \alpha^2}{\alpha(1 + \alpha^2)} \int_0^\infty \frac{2(1 + \alpha^2)}{(1 + \alpha^2 - 2\alpha) + (1 + \alpha^2 + 2\alpha)t^2} dt \\
&= \frac{\pi}{\alpha} - \frac{1 - \alpha^2}{\alpha(1 + \alpha^2)} \int_0^\infty \frac{2(1 + \alpha^2)}{(1 - \alpha)^2 + (1 + \alpha)^2 t^2} dt = \frac{\pi}{\alpha} - \frac{2(1 + \alpha)}{\alpha(1 - \alpha)} \int_0^\infty \frac{1}{1 + \left(\frac{\alpha + 1}{\alpha - 1} \right)^2 t^2} dt \\
&= \frac{\pi}{\alpha} - \frac{2(1 + \alpha)}{\alpha(1 - \alpha)} \left[\frac{\alpha - 1}{\alpha + 1} \tan^{-1} \left(\frac{\alpha + 1}{\alpha - 1} t \right) \right]_0^\infty = \frac{\pi}{\alpha} + \frac{\pi}{\alpha} = \frac{2\pi}{\alpha}
\end{aligned}$$

$$I(\alpha) = \int \frac{2\pi}{\alpha} d\alpha = 2\pi \ln |\alpha| + C, \quad I(1) = \int_0^\pi \ln(2 - 2\cos x) dx = 0 = C \quad (*)$$

$$I(\alpha) = \int_0^\pi \ln(1 - 2\alpha \cos x + \alpha^2) dx = 2\pi \ln |\alpha|$$

$$\therefore \int_0^\pi \ln(1 - 2e \cos x + e^2) dx = I(e) = 2\pi \quad \blacksquare$$

$$* \quad J = \int_0^\pi \ln(2 - 2\cos x) dx = \pi \ln 2 + \int_0^\pi \ln(1 - \cos x) dx$$

$$\begin{aligned}
&= \pi \ln 2 + \int_0^\pi \ln(1 + \cos x) dx \quad (x \mapsto \pi - x) \\
&= \pi \ln 2 + \frac{1}{2} \int_0^\pi \ln(1 - \cos^2 x) dx = \pi \ln 2 + \frac{1}{2} \int_0^\pi \ln(\sin^2 x) dx \\
&= \pi \ln 2 + \int_0^\pi \ln(\sin x) dx = \pi \ln 2 + 2K \\
K &= \frac{1}{2} \int_0^\pi \ln(\sin x) dx = \int_0^{\frac{\pi}{2}} \ln(\sin x) dx = \int_0^{\frac{\pi}{2}} \ln(\cos x) dx \quad (x \mapsto \frac{\pi}{2} - x) \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln(\sin x \cos x) dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{2} \sin 2x\right) dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln(\sin 2x) dx - \frac{\pi}{4} \ln 2 \\
&= \frac{1}{4} \int_0^\pi \ln(\sin t) dt - \frac{\pi}{4} \ln 2 \quad (t = 2x, \quad dt = 2dx) \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln(\sin t) dt - \frac{\pi}{4} \ln 2 = \frac{1}{2} I - \frac{\pi}{4} \ln 2, \quad \frac{1}{2} I = -\frac{\pi}{4} \ln 2 \\
\therefore K &= \int_0^{\frac{\pi}{2}} \ln(\sin x) dx = -\frac{\pi}{2} \ln 2 \\
\therefore J &= \pi \ln 2 + 2K = 0 \quad \blacksquare
\end{aligned}$$

55. $x = \tan t, \quad dx = \sec^2 t dt$

$$\begin{aligned}
I &= \int_0^\pi \ln(1 + \sin^2 t) dt = 2 \int_0^{\frac{\pi}{2}} \ln(1 + \sin^2 t) dt = 2 \int_0^\infty \frac{\ln(2 + x^2) - \ln(1 + x^2)}{1 + x^2} dx \\
I(\alpha) &:= \int_0^\infty \frac{\ln(\alpha + x^2) - \ln(1 + x^2)}{1 + x^2} dx, \\
\frac{d}{d\alpha} I(\alpha) &= \int_0^\infty \frac{\partial}{\partial \alpha} \frac{\ln(\alpha + x^2) - \ln(1 + x^2)}{1 + x^2} dx = \int_0^\infty \frac{1}{(1 + x^2)(\alpha + x^2)} dx \\
&= \frac{1}{\alpha - 1} \left(\int_0^\infty \frac{1}{x^2 + 1} dx - \int_0^\infty \frac{1}{x^2 + \alpha} dx \right) = \frac{1}{\alpha - 1} \left([\tan^{-1} x]_0^\infty - \left[\frac{1}{\sqrt{\alpha}} \tan^{-1} \left(\frac{x}{\sqrt{\alpha}} \right) \right]_0^\infty \right)
\end{aligned}$$

$$= \frac{\pi}{2\sqrt{\alpha}(1+\sqrt{\alpha})}$$

한편 $I(1) = \int_0^{\infty} 0 dx = 0$ 이므로

$$I = 2I(2) = 2 \int_1^2 \frac{d}{d\alpha} I(\alpha) d\alpha = \pi \int_1^2 \frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})} d\alpha$$

$$= 2\pi \int_1^{\sqrt{2}} \frac{1}{1+u} du \quad (\alpha = u^2, d\alpha = 2udu)$$

$$= 2\pi [\ln|1+u|]_1^{\sqrt{2}} = 2\pi \ln\left(\frac{1+\sqrt{2}}{2}\right) \blacksquare$$

56. $x = u + 2, dx = du$

$$I = \int_0^4 \frac{\ln x}{\sqrt{4x-x^2}} dx = \int_{-2}^2 \frac{\ln(u+2)}{\sqrt{4-u^2}} du = \int_0^2 \frac{\ln(u+2)}{\sqrt{4-u^2}} du + \int_{-2}^0 \frac{\ln(u+2)}{\sqrt{4-u^2}} du$$

$$= \int_0^2 \frac{\ln(u+2)}{\sqrt{4-u^2}} du + \int_2^0 \frac{\ln(-u+2)}{\sqrt{4-u^2}} (-du) \quad (u \mapsto -u)$$

$$= \int_0^2 \frac{\ln(4-u^2)}{\sqrt{4-u^2}} du$$

한편 $x = 4 - u^2, dx = -2udu$ 의 치환을 하면

$$I = \int_0^4 \frac{\ln x}{\sqrt{4x-x^2}} dx = \int_2^0 \frac{\ln(4-u^2)}{u\sqrt{4-u^2}} \cdot (-2udu) = 2 \int_0^2 \frac{\ln(4-u^2)}{\sqrt{4-u^2}} du$$

$$I = 2I, \therefore I = \int_0^4 \frac{\ln x}{\sqrt{4x-x^2}} dx = 0 \blacksquare$$

57. $x = \tan t, dx = \sec^2 t dt, \frac{dx}{1+x^2} = dt$

$$I = \int_0^{\infty} \frac{1}{(1+x^2)(1+x^\pi)} dx = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^\pi t} dt$$

$$= \int_{\frac{\pi}{2}}^0 \frac{1}{1 + \tan^{\pi}\left(\frac{\pi}{2} - t\right)} (-dt) = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \cot^{\pi}t} dt = \int_0^{\frac{\pi}{2}} \frac{\tan^{\pi}t}{1 + \tan^{\pi}t} dt \quad (t \mapsto \frac{\pi}{2} - t)$$

$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{1}{1 + \tan^{\pi}t} + \frac{\tan^{\pi}t}{1 + \tan^{\pi}t} \right) dt = \frac{\pi}{2}$$

$$\therefore I = \int_0^{\infty} \frac{1}{(1+x^2)(1+x^{\pi})} dx = \frac{\pi}{4} \blacksquare$$

58. sol 1) $I(v) := \int_0^{\infty} \frac{1}{(v+x)(\pi^2 + (\ln x)^2)} dx$, $J(v) := \int_{-\infty}^{\infty} \frac{1}{(v+e^t)(\pi^2 + t^2)} dt$

$$x = e^t, \quad dx = e^t dt$$

$$I(v) = \int_{-\infty}^{\infty} \frac{e^t}{(v+e^t)(\pi^2 + t^2)} dt = \int_{-\infty}^{\infty} \frac{(v+e^t)}{(v+e^t)(\pi^2 + t^2)} dt - v \int_{-\infty}^{\infty} \frac{1}{(v+e^t)(\pi^2 + t^2)} dt$$

$$= \left[\frac{1}{\pi} \tan^{-1}\left(\frac{x}{\pi}\right) \right]_{-\infty}^{\infty} - v J(v) = 1 - v J(v)$$

$$t = -z, \quad dt = -dz$$

$$I(v) = \int_{\infty}^{-\infty} \frac{e^{-z}}{(v+e^{-z})(\pi^2 + z^2)} (-dz) = \int_{-\infty}^{\infty} \frac{1}{(1+ve^z)(\pi^2 + z^2)} dz = \frac{1}{v} J\left(\frac{1}{v}\right)$$

$$1 - v J(v) = I(v) = \frac{1}{v} J\left(\frac{1}{v}\right), \quad v J(v) + \frac{1}{v} J\left(\frac{1}{v}\right) = 1$$

$$v = 1, \quad \therefore J(1) = I(1) = \int_0^{\infty} \frac{1}{(1+x)(\pi^2 + (\ln x)^2)} dx = \frac{1}{2} \blacksquare$$

sol 2) $I = \int_0^{\infty} \frac{1}{(1+x)(\pi^2 + (\ln x)^2)} dx$

$$= \int_0^1 \frac{1}{(1+x)(\pi^2 + (\ln x)^2)} dx + \int_1^{\infty} \frac{1}{(1+x)(\pi^2 + (\ln x)^2)} dx$$

$$= \int_0^1 \frac{1}{(1+x)(\pi^2 + (\ln x)^2)} dx + \int_1^0 \frac{1}{(1+1/x)(\pi^2 + (\ln x)^2)} \cdot \left(-\frac{1}{x^2}\right) dx \quad (x \mapsto \frac{1}{x})$$

$$\begin{aligned}
&= \int_0^1 \frac{1}{(1+x)(\pi^2 + (\ln x)^2)} dx + \int_0^1 \frac{1}{x(1+x)(\pi^2 + (\ln x)^2)} dx \\
&= \int_0^1 \left(\frac{1}{(1+x)(\pi^2 + (\ln x)^2)} \cdot \frac{x+1}{x} \right) dx = \int_0^1 \frac{1}{x(\pi^2 + (\ln x)^2)} dx \\
&= \int_{-\infty}^0 \frac{1}{\pi^2 + u^2} du \quad (u = \ln x, \quad du = \frac{dx}{x}) \\
&= \left[\frac{1}{\pi} \tan^{-1} \left(\frac{u}{\pi} \right) \right]_{-\infty}^0 = \frac{1}{2} \blacksquare
\end{aligned}$$

59. $I = \int_0^\infty \frac{x-1}{\sqrt{2^x-1} \ln(2^x-1)} dx$

$$\begin{aligned}
&= \frac{1}{(\ln 2)^2} \int_0^\infty \frac{\ln(t^2+1) - \ln 2}{(t^2+1)\ln t} dt \quad (t = \sqrt{2^x-1}, \quad dt = \frac{2^{x-1} \ln 2}{\sqrt{2^x-1}} dx) \\
&= \frac{1}{(\ln 2)^2} \int_\infty^0 \frac{\ln(u^{-2}+1) - \ln 2}{\left(\frac{1}{u^2}+1\right)(-\ln u)} \left(-\frac{1}{u^2}\right) du = -\frac{1}{(\ln 2)^2} \int_0^\infty \frac{\ln(u^2+1) - \ln u^2 - \ln 2}{(u^2+1)\ln u} du \\
&(u = \frac{1}{t}, \quad dt = -\frac{1}{u^2} du) \\
&= -\frac{1}{(\ln 2)^2} \int_0^\infty \frac{\ln(u^2+1) - \ln 2}{(u^2+1)\ln u} du + \frac{2}{(\ln 2)^2} \int_0^\infty \frac{1}{u^2+1} du \\
&= -I + \frac{2}{(\ln 2)^2} [\tan^{-1} u]_0^\infty = -I + \frac{\pi}{(\ln 2)^2} \\
\therefore I &= \int_0^\infty \frac{x-1}{\sqrt{2^x-1} \ln(2^x-1)} dx = \frac{\pi}{2(\ln 2)^2} \blacksquare
\end{aligned}$$

60. $t = \tan x, \quad dt = \sec^2 x dx$

$$\begin{aligned}
\int_0^{3\pi} \frac{1}{\sin^4 x + \cos^4 x} dx &= 6 \int_0^{\frac{\pi}{2}} \frac{\sec^4 x}{1 + \tan^4 x} dx = 6 \int_0^\infty \frac{1+t^2}{1+t^4} dt \\
&= 6 \int_0^\infty \left(\frac{1}{2(t^2 + \sqrt{2}t + 1)} - \frac{1}{2(-t^2 + \sqrt{2}t - 1)} \right) dt
\end{aligned}$$

$$\begin{aligned}
&= 3 \int_0^\infty \frac{1}{t^2 + \sqrt{2}t + 1} dt + 3 \int_0^\infty \frac{1}{t^2 - \sqrt{2}t + 1} dt \\
&= 3 \int_0^\infty \frac{1}{\left(t + \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} dt + 3 \int_0^\infty \frac{1}{\left(t - \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} dt \\
&= 3 \left[\sqrt{2} \tan^{-1}(\sqrt{2}t + 1) \right]_0^\infty + 3 \left[\sqrt{2} \tan^{-1}(\sqrt{2}t - 1) \right]_0^\infty = 3\sqrt{2}\pi \blacksquare
\end{aligned}$$

$$\begin{aligned}
61. \int_0^\pi \cos^4 x dx &= \frac{1}{4} \int_0^\pi (1 - \cos 2x)^2 dx = \frac{1}{4} \int_0^\pi \left(1 - 2\cos 2x + \frac{1 + \cos 4x}{2}\right) dx \\
&= \int_0^\pi \left(\frac{3}{8} - \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x\right) dx = \left[\frac{3}{8}x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x\right]_0^\pi = \frac{3}{8}\pi \blacksquare
\end{aligned}$$

$$\begin{aligned}
62. \int_0^{\frac{\pi}{2}} \sin^6 x \cos^3 x dx &= \int_0^{\frac{\pi}{2}} \sin^6 x (1 - \sin^2 x) \cos x dx \\
&= \int_0^{\frac{\pi}{2}} \sin^6 x \cos x dx - \int_0^{\frac{\pi}{2}} \sin^8 x \cos x dx = \left[\frac{1}{7} \sin^7 x - \frac{1}{9} \sin^9 x\right]_0^{\frac{\pi}{2}} = \frac{2}{63} \blacksquare
\end{aligned}$$

$$\begin{aligned}
63. \int_0^{\frac{\pi}{4}} \sin^3 x \sec^2 x dx &= \int_0^{\frac{\pi}{4}} \frac{\sin x (1 - \cos^2 x)}{\cos^2 x} dx = \int_0^{\frac{\pi}{4}} (\tan x \sec x - \sin x) dx \\
&= [\sec x + \cos x]_0^{\frac{\pi}{4}} = \frac{3\sqrt{2}}{2} - 2 \blacksquare
\end{aligned}$$

$$64. x = 2 \tan t, \quad dx = 2 \sec^2 t dt$$

$$\begin{aligned}
\int_1^3 \frac{1}{x^2 \sqrt{x^2 + 4}} dx &= \int_\alpha^\beta \frac{\sec t}{4 \tan^2 t} dt = \frac{1}{4} \int_\alpha^\beta \csc t \cot t dt = \left[-\frac{1}{4} \csc t\right]_\alpha^\beta = \left[-\frac{\sqrt{x^2 + 4}}{4x}\right]_1^3 \\
&= \frac{3\sqrt{5} - \sqrt{13}}{12} \blacksquare \quad \left(\alpha = \tan^{-1}\left(\frac{1}{2}\right), \beta = \tan^{-1}\left(\frac{3}{2}\right)\right)
\end{aligned}$$

$$65. x + 1 = \tan t, \quad dx = \sec^2 t dt$$

$$\int_{\sqrt{3}-1}^{2\sqrt{2}-1} \frac{1}{(x+1)\sqrt{x^2+2x+2}} dx = \int_{\alpha}^{\beta} \frac{\sec^2 t}{\tan t \sec t} dt = \int_{\alpha}^{\beta} \csc t dt = [\ln|\csc t - \cot t|]_{\alpha}^{\beta}$$

$$= \left[\ln \left| \frac{\sqrt{x^2+2x+2}-1}{x+1} \right| \right]_{\sqrt{3}-1}^{2\sqrt{2}-1} = \ln \sqrt{\frac{3}{2}} \quad \blacksquare \quad (\alpha = \frac{\pi}{3}, \beta = \tan^{-1}(2\sqrt{2}))$$

$$66. \int_0^{\tan^{-1}(\sqrt{6})} \frac{1}{1+\cos^2 x} dx = \int_0^{\tan^{-1}(\sqrt{6})} \sec^2 x \cdot \frac{1}{\tan^2 x + 2} dx$$

$$= \left[\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\tan x}{\sqrt{2}} \right) \right]_0^{\tan^{-1}(\sqrt{6})} = \frac{\sqrt{2}}{6} \pi \quad \blacksquare$$

$$67. \text{ sol 1) } \int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x} dx = \int_0^{\frac{\pi}{2}} \frac{1-\cos x}{\sin^2 x} dx = \int_0^{\frac{\pi}{2}} (\csc^2 x - \csc x \cot x) dx$$

$$= [\csc x - \cot x]_0^{\frac{\pi}{2}} = 1 - \lim_{x \rightarrow 0} (\csc x - \cot x) = 1 \quad \blacksquare$$

$$\text{sol 2) } \int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x} dx = \int_0^{\frac{\pi}{2}} \frac{1}{2\cos^2 \frac{x}{2}} dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \sec^2 \frac{x}{2} dx = \left[\tan \frac{x}{2} \right]_0^{\frac{\pi}{2}} = 1 \quad \blacksquare$$

$$68. t = e^x + 1, dt = e^x dx$$

$$\int_{\ln(e-1)}^{\ln(e^3-1)} \frac{e^x \ln(e^x+1)}{e^x+1} dx = \int_e^{e^3} \frac{\ln t}{t} dt = \left[\frac{1}{2} (\ln t)^2 \right]_e^{e^3} = 4 \quad \blacksquare$$

$$69. \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{\sin x} dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \csc x dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x} dx$$

$$= [\ln|\csc x - \cot x|]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \ln \sqrt{3} \quad \blacksquare$$

$$70. \int_0^{\frac{\pi}{4}} \frac{1}{1-3\cos^2 x} dx = \int_0^{\frac{\pi}{4}} \sec^2 x \cdot \frac{1}{\tan^2 x - 2} dx = \left[\frac{1}{2\sqrt{2}} \ln \left| \frac{\tan x - \sqrt{2}}{\tan x + \sqrt{2}} \right| \right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2\sqrt{2}} \ln(3-2\sqrt{2}) = \frac{1}{\sqrt{2}} \ln(\sqrt{2}-1) \quad \blacksquare$$

$$71. \int_{-1}^0 \frac{x^3 - x - 2}{x^3 - x^2 + x - 1} dx = \int_{-1}^0 \left(1 - \frac{1}{x-1} + \frac{2x}{x^2+1} \right) dx = [x - \ln|x-1| + \ln(x^2+1)]_{-1}^0$$

$$= 1 \blacksquare$$

$$72. \int_3^4 \frac{x+4}{x^3+3x^2-10x} dx = \int_3^4 \frac{x+4}{x(x+5)(x-2)} dx = \int_3^4 \left(-\frac{2}{5x} - \frac{1}{35(x+5)} + \frac{3}{7(x-2)} \right) dx$$

$$= \left[-\frac{2}{5} \ln|x| - \frac{1}{35} \ln|x+5| + \frac{3}{7} \ln|x-2| \right]_3^4 = \frac{12}{35} \ln 3 - \frac{10}{35} \ln 2 = \frac{2}{35} \ln \left(\frac{729}{32} \right) \blacksquare$$

$$73. \int_5^6 \frac{7x^3 - 13x^2 - 24x + 24}{x^4 - 3x^3 - 10x^2 + 24x} dx = \int_5^6 \frac{7x^3 - 13x^2 - 24x + 24}{x(x-2)(x+3)(x-4)} dx$$

$$= \int_5^6 \left(\frac{1}{x-2} + \frac{1}{x} + \frac{2}{x+3} + \frac{3}{x-4} \right) dx = [\ln|x-2| + \ln|x| + 2\ln|x+3| + 3\ln|x-4|]_5^6$$

$$= 4\ln 3 - \ln 5 = \ln \left(\frac{81}{5} \right) \blacksquare$$

$$74. t = \frac{1}{x}, \quad dt = -\frac{1}{x^2} dx$$

$$\int_1^2 \frac{1}{x^2 \sqrt{2x-x^2}} dx = \int_1^2 \frac{1}{x^3 \sqrt{\frac{2}{x}-1}} dx = -\int_1^{\frac{1}{2}} \frac{t}{\sqrt{2t-1}} dt = \frac{1}{2} \int_{\frac{1}{2}}^1 \frac{2t-1+1}{\sqrt{2t-1}} dt$$

$$= \frac{1}{2} \int_{\frac{1}{2}}^1 \left(\sqrt{2t-1} + \frac{1}{\sqrt{2t-1}} \right) dt = \frac{1}{2} \left[\frac{1}{6} (2t-1)^{\frac{3}{2}} + \frac{1}{2} \sqrt{2t-1} \right]_{\frac{1}{2}}^1 = \frac{2}{3} \blacksquare$$

$$75. t = x^2, \quad dt = 2x dx$$

$$\int_1^e \frac{x^4 + 81}{x(x^2+9)^2} dx = \frac{1}{2} \int_1^{e^2} \frac{t^2 + 81}{t(t+9)^2} dt = \frac{1}{2} \int_1^{e^2} \left(\frac{1}{t} - \frac{18}{(t+9)^2} \right) dt$$

$$= \left[\frac{1}{2} \ln|t| + \frac{9}{t+9} \right]_1^{e^2} = \frac{e^2 + 99}{10(e^2 + 9)} \blacksquare$$

$$76. t = \sqrt{x}, \quad dt = \frac{1}{2\sqrt{x}} dx, \quad t = 2\sin\theta, \quad dt = 2\cos\theta d\theta, \quad \sin\theta = \frac{\sqrt{x}}{2}$$

$$\int_0^3 \sqrt{\frac{4-x}{x}} dx = \int_0^{\sqrt{3}} \frac{\sqrt{4-t^2}}{t} \cdot 2t dt = 2 \int_0^{\sqrt{3}} \sqrt{4-t^2} dt = 2 \int_0^{\frac{\pi}{3}} 4 \cos^2 \theta d\theta$$

$$= 4 \int_0^{\frac{\pi}{3}} (1 + \cos 2\theta) d\theta = [4\theta + 4 \sin \theta \cos \theta]_0^{\frac{\pi}{3}} = \sqrt{3} + \frac{4}{3} \pi \blacksquare$$

77. $t = x^{\frac{3}{2}}, dt = \frac{3}{2} \sqrt{x} dx$

$$\int_0^{\frac{1}{\sqrt{2}}} \sqrt{\frac{x}{1-x^3}} dx = \frac{2}{3} \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-t^2}} dt = \left[\frac{2}{3} \sin^{-1} t \right]_0^{\frac{1}{\sqrt{2}}} = \frac{\pi}{6} \blacksquare$$

78. $t = \sqrt{x}, dt = \frac{1}{2\sqrt{x}} dx, t = \sin u, dt = \cos u du$

$$\int_0^{\frac{3}{4}} \sqrt{x} \sqrt{1-x} dx = \int_0^{\frac{\sqrt{3}}{2}} t \sqrt{1-t^2} \cdot 2t dt = 2 \int_0^{\frac{\sqrt{3}}{2}} t^2 \sqrt{1-t^2} dt$$

$$= 2 \int_0^{\frac{\pi}{3}} \sin^2 u \cos u \cdot \cos u du = \frac{1}{2} \int_0^{\frac{\pi}{3}} \sin^2 2u du = \frac{1}{4} \int_0^{\frac{\pi}{3}} (1 - \cos 4u) du$$

$$= \left[\frac{1}{4} u - \frac{1}{16} \sin 4u \right]_0^{\frac{\pi}{3}} = \frac{\sqrt{3}}{32} + \frac{\pi}{12} \blacksquare$$

79. $t = \tan x, dt = \sec^2 x dx$

$$\int_0^{\tan^{-1}(\sqrt[4]{3})} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx = \int_0^{\tan^{-1}(\sqrt[4]{3})} \frac{\tan x \sec^2 x}{1 + \tan^4 x} dx = \int_0^{\sqrt[4]{3}} \frac{t}{1+t^4} dt$$

$$= \left[\frac{1}{2} \tan^{-1}(t^2) \right]_0^{\sqrt[4]{3}} = \frac{\pi}{6} \blacksquare$$

80. $\int_0^{\frac{\pi}{4}} \tan^4 x \sec^4 x dx = \int_0^{\frac{\pi}{4}} \tan^4 x (\tan^2 x + 1) \sec^2 x dx$

$$= \int_0^{\frac{\pi}{4}} \tan^6 x \sec^2 x dx + \int_0^{\frac{\pi}{4}} \tan^4 x \sec^2 x dx = \left[\frac{1}{7} \tan^7 x + \frac{1}{5} \tan^5 x \right]_0^{\frac{\pi}{4}} = \frac{12}{35} \blacksquare$$

$$81. \quad t = \sqrt{x}, \quad dt = \frac{1}{2\sqrt{x}} dx$$

$$\begin{aligned} \int_2^4 \frac{x\sqrt{x}-1}{x-\sqrt{x}} dx &= \int_{\sqrt{2}}^2 \frac{t^3-1}{t^2-t} \cdot 2t dt = 2 \int_{\sqrt{2}}^2 (t^2+t+1) dt = \left[\frac{2}{3}t^3 + t^2 + 2t \right]_{\sqrt{2}}^2 \\ &= \frac{34-10\sqrt{2}}{3} \blacksquare \end{aligned}$$

$$82. \quad t = e^x, \quad dt = e^x dx, \quad u = \sqrt{t+1}, \quad du = \frac{1}{2\sqrt{t+1}} dt$$

$$\begin{aligned} \int_0^{\ln 3} \frac{1}{\sqrt{1+e^x}} dx &= \int_1^3 \frac{1}{t\sqrt{t+1}} dt = \int_{\sqrt{2}}^2 \frac{2}{u^2-1} du = \left[\ln \left| \frac{u-1}{u+1} \right| \right]_{\sqrt{2}}^2 \\ &= -\ln(9-6\sqrt{2}) \blacksquare \end{aligned}$$

$$83. \quad \int_1^2 \frac{x^2+2x-1}{2x^3+3x^2-2x} dx = \int_1^2 \frac{x^2+2x-1}{x(x+2)(2x-1)} dx = \int_1^2 \left(-\frac{1}{10(x+2)} + \frac{1}{5(2x-1)} + \frac{1}{2x} \right) dx$$

$$= \left[-\frac{1}{10} \ln|x+2| + \frac{1}{10} \ln|2x-1| + \frac{1}{2} \ln|x| \right]_1^2 = \frac{3}{10} \ln 2 + \frac{1}{5} \ln 3 = \frac{1}{10} \ln 72 \blacksquare$$

$$84. \quad I = \int_0^{\frac{\pi}{3}} 13e^{2x} \cos 3x dx = \left[\frac{13}{2} e^{2x} \cos 3x \right]_0^{\frac{\pi}{3}} + \frac{3}{2} \int_0^{\frac{\pi}{3}} 13e^{2x} \sin 3x dx$$

$$= -\frac{13}{2}(e^{2\pi/3} + 1) + \frac{39}{2} \left\{ \left[\frac{1}{2} e^{2x} \sin 3x \right]_0^{\frac{\pi}{3}} - \frac{1}{2} \int_0^{\frac{\pi}{3}} 3e^{2x} \cos 3x dx \right\}$$

$$= -\frac{13}{2}(e^{2\pi/3} + 1) - \frac{9}{4} I$$

$$\frac{13}{4} I = -\frac{13}{2}(e^{2\pi/3} + 1)$$

$$\therefore I = \int_0^{\frac{\pi}{3}} 13e^{2x} \cos 3x dx = -2(e^{2\pi/3} + 1) \blacksquare$$

$$85. \quad \int_0^{\pi} e^{-x} \sin^2 2x dx = \frac{1}{2} \int_0^{\pi} e^{-x} (1 - \cos 4x) dx = \frac{1}{2} [-e^{-x}]_0^{\pi} - \frac{1}{2} \int_0^{\pi} e^{-x} \cos 4x dx$$

$$\begin{aligned}
&= \frac{1-e^{-\pi}}{2} - \frac{1}{2} \left(\left[\frac{1}{4} e^{-x} \sin 4x \right]_0^{\pi} + \frac{1}{4} \int_0^{\pi} e^{-x} \sin 4x dx \right) \\
&= \frac{1-e^{-\pi}}{2} - \frac{1}{8} \left(\left[-\frac{1}{4} e^{-x} \cos 4x \right]_0^{\pi} - \frac{1}{4} \int_0^{\pi} e^{-x} \cos 4x dx \right) \\
&= \frac{15(1-e^{-\pi})}{32} + \frac{1}{32} \int_0^{\pi} e^{-x} \cos 4x dx \\
\frac{17}{32} \int_0^{\pi} e^{-x} \cos 4x dx &= \frac{1-e^{-\pi}}{32}, \quad \int_0^{\pi} e^{-x} \cos 4x dx = \frac{1-e^{-\pi}}{17} \\
\therefore \int_0^{\pi} e^{-x} \sin^2 2x dx &= \frac{1-e^{-\pi}}{2} - \frac{1}{2} \int_0^{\pi} e^{-x} \cos 4x dx = \frac{8}{17} (1-e^{-\pi}) \blacksquare
\end{aligned}$$

86. sol 1) $t^2 = \tan x$, $2t dt = \sec^2 x dx$, $dx = \frac{2t}{t^4+1} dt$

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \sqrt{\tan x} dx &= \int_0^1 \frac{2t^2}{t^4+1} dt = \int_0^1 \frac{2}{t^2+\frac{1}{t^2}} dt = \int_0^1 \frac{\left(1+\frac{1}{t^2}\right) + \left(1-\frac{1}{t^2}\right)}{t^2+\frac{1}{t^2}} dt \\
&= \int_0^1 \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}} dt + \int_0^1 \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}} dt = \int_0^1 \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+(\sqrt{2})^2} dt + \int_0^1 \frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2-(\sqrt{2})^2} dt \\
&= \left[\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{t-1/t}{\sqrt{2}} \right) + \frac{1}{2\sqrt{2}} \ln \left| \frac{t+1/t-\sqrt{2}}{t+1/t+\sqrt{2}} \right| \right]_0^1 = \frac{\pi}{2\sqrt{2}} + \frac{1}{\sqrt{2}} \ln(\sqrt{2}-1) \blacksquare
\end{aligned}$$

sol 2) $I := \int_0^{\frac{\pi}{4}} \sqrt{\tan x} dx$, $J := \int_0^{\frac{\pi}{4}} \sqrt{\cot x} dx$

$$\begin{aligned}
I+J &= \int_0^{\frac{\pi}{4}} (\sqrt{\tan x} + \sqrt{\cot x}) dx = \sqrt{2} \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx \\
&= \sqrt{2} \int_0^{\frac{\pi}{4}} \frac{(\sin x - \cos x)'}{\sqrt{1 - (\sin x - \cos x)^2}} dx = \sqrt{2} [\sin^{-1}(\sin x - \cos x)]_0^{\frac{\pi}{4}} = \frac{\pi}{\sqrt{2}} \dots [1]
\end{aligned}$$

$$\begin{aligned}
I - J &= \int_0^{\frac{\pi}{4}} (\sqrt{\tan x} - \sqrt{\cot x}) dx = \sqrt{2} \int_0^{\frac{\pi}{4}} \frac{\sin x - \cos x}{\sqrt{\sin 2x}} dx \\
&= -\sqrt{2} \int_0^{\frac{\pi}{4}} \frac{(\sin x + \cos x)'}{\sqrt{(\sin x + \cos x)^2 - 1}} dx \\
&= \left[-\sqrt{2} \ln \left| (\sin x + \cos x) + \sqrt{(\sin x + \cos x)^2 - 1} \right| \right]_0^{\frac{\pi}{4}} \\
&= -\sqrt{2} \ln(\sqrt{2} + 1) = \sqrt{2} \ln(\sqrt{2} - 1) \quad \dots \quad [2]
\end{aligned}$$

$$\frac{[1] + [2]}{2} : \therefore I = \int_0^{\frac{\pi}{4}} \sqrt{\tan x} dx = \frac{\pi}{2\sqrt{2}} + \frac{1}{\sqrt{2}} \ln(\sqrt{2} - 1) \quad \blacksquare$$

87. $t = \tan x$, $dt = \sec^2 x dx$, $u = \frac{t}{1-t}$, $du = \frac{1}{(1-t)^2} dt$, $u = y^2$, $du = 2y dy$

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{4}} \sqrt{\tan x} \sqrt{1 - \tan x} dx = \int_0^1 \frac{\sqrt{t(1-t)}}{1+t^2} dt = \int_0^\infty \frac{\sqrt{u}}{(1+u)(1+2u+2u^2)} du \\
&= \int_0^\infty \frac{2y^2}{(1+y^2)(1+2y^2+2y^4)} dy = 2 \int_0^\infty \left(\frac{2y^2}{1+2y^2+2y^4} + \frac{1}{1+2y^2+2y^4} - \frac{1}{1+y^2} \right) dy \\
&= \int_{-\infty}^\infty \left(\frac{2y^2}{1+2y^2+2y^4} + \frac{1}{1+2y^2+2y^4} - \frac{1}{1+y^2} \right) dy = I_1 + I_2 - [\tan^{-1} y]_{-\infty}^\infty = I_1 + I_2 - \pi
\end{aligned}$$

$$I_1 = \int_{-\infty}^\infty \frac{2y^2}{1+2y^2+2y^4} dy = \int_{-\infty}^\infty \frac{1}{y^2 + \frac{1}{2y^2} + 1} dy = \int_{-\infty}^\infty \frac{1}{\left(y - \frac{1}{\sqrt{2}y}\right)^2 + 1 + \sqrt{2}} dy$$

$$= \int_{-\infty}^\infty \frac{1}{y^2 + 1 + \sqrt{2}} dy \quad (\because \text{Glasser's Master Theorem})$$

$$= \left[\frac{1}{\sqrt{1+\sqrt{2}}} \tan^{-1} \left(\frac{y}{\sqrt{1+\sqrt{2}}} \right) \right]_{-\infty}^\infty = \frac{\pi}{\sqrt{1+\sqrt{2}}}$$

$$I_2 = \frac{1}{2} \int_{-\infty}^\infty \frac{1}{y^4 + y^2 + \frac{1}{2}} dy = \int_0^\infty \frac{1}{y^2 + \frac{1}{2y^2} + 1} \cdot \frac{dy}{y^2} = \int_0^\infty \frac{1}{\frac{y^2}{2} + \frac{1}{y^2} + 1} dy \quad (y \mapsto \frac{1}{y})$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \frac{1}{y^2 + \frac{2}{y^2} + 2} dy = \int_{-\infty}^{\infty} \frac{1}{\left(y - \frac{\sqrt{2}}{y}\right)^2 + 2 + 2\sqrt{2}} dy \\
&= \int_{-\infty}^{\infty} \frac{1}{y^2 + 2 + 2\sqrt{2}} dy \quad (\because \text{Glasser's Master Theorem}) \\
&= \left[\frac{1}{\sqrt{2+2\sqrt{2}}} \tan^{-1}\left(\frac{y}{\sqrt{2+2\sqrt{2}}}\right) \right]_{-\infty}^{\infty} = \frac{\pi}{\sqrt{2+2\sqrt{2}}} \\
\therefore I &= \int_0^{\frac{\pi}{4}} \sqrt{\tan x} \sqrt{1 - \tan x} dx = I_1 + I_2 - \pi = \left(\sqrt{\frac{1+\sqrt{2}}{2}} - 1 \right) \pi \blacksquare
\end{aligned}$$

$$\begin{aligned}
88. \quad I_{m,n} &:= \int_1^{\infty} \frac{(\ln x)^m}{x^n} dx = \int_1^{\infty} \frac{(\ln x)^m}{x} \cdot \frac{1}{x^{n-1}} dx \\
&= \left[\frac{(\ln x)^{m+1}}{(m+1)x^{n-1}} \right]_1^{\infty} - \frac{1-n}{m+1} \int_1^{\infty} \frac{(\ln x)^{m+1}}{x^n} dx = \frac{n-1}{m+1} \int_1^{\infty} \frac{(\ln x)^{m+1}}{x^n} dx \\
&= \frac{n-1}{m+1} I_{m+1,n}, \quad I_{m,n} = \frac{m}{n-1} I_{m-1,n} \text{을 얻는다. 한편}
\end{aligned}$$

$$I_{0,n} = \int_1^{\infty} \frac{1}{x^n} dx = \frac{1}{n-1} \text{이므로}$$

$$I_{m,n} = \frac{m!}{(n-1)^{m+1}}, \quad m, n \in \mathbb{N}_0.$$

$$\therefore \int_1^{\infty} \frac{(\ln x)^{627}}{x^{2022}} dx = I_{627,2022} = \frac{627!}{2021^{628}} \blacksquare$$

$$89. \quad x = \cot t, \quad \sin(\cot^{-1} x) = \sin t = \frac{x}{\sqrt{1+x^2}}, \quad x = \tan u, \quad \cos(\tan^{-1} x) = \cos u = \frac{1}{\sqrt{1+x^2}}$$

$$\begin{aligned}
&\int_0^{\sqrt{6}} \cos^2(\tan^{-1}(\sin(\cot^{-1} x))) dx = \int_0^{\sqrt{6}} \cos^2\left(\tan^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right) dx \\
&= \int_0^{\sqrt{6}} \frac{1+x^2}{2+x^2} dx = \int_0^{\sqrt{6}} \left(1 - \frac{1}{2+x^2}\right) dx = \left[x - \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) \right]_0^{\sqrt{6}} = \sqrt{6} - \frac{\pi}{3\sqrt{2}} \blacksquare
\end{aligned}$$

90. $t = \tan x$, $dt = \sec^2 x dx$, $u = t + \sqrt{1+t^2}$, $t = \frac{1}{2}\left(u - \frac{1}{u}\right)$, $dt = \frac{1}{2}\left(1 + \frac{1}{u^2}\right)du$

$$\int_0^{\frac{\pi}{6}} \frac{\sec^2 x}{(\sec x + \tan x)^{5/2}} dx = \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{(t + \sqrt{1+t^2})^{5/2}} dt = \frac{1}{2} \int_1^{\sqrt{3}} (u^{-5/2} + u^{-9/2}) du$$

$$= \left[-\frac{1}{3} u^{-3/2} - \frac{1}{7} u^{-7/2} \right]_1^{\sqrt{3}} = -\frac{8}{63} \sqrt[4]{3} + \frac{10}{21} \blacksquare$$

91. 제 1종 오일러 치환 : $\sqrt{x^2+4x-4} = x+t$, $x = \frac{t^2+4}{4-2t}$, $dx = \frac{-2t^2+8t+8}{(4-2t)^2} dt$

$$\int_1^2 \frac{1}{x\sqrt{x^2+4x-4}} dx = \int_0^{2\sqrt{2}-2} \frac{4-2t}{t^2+4} \cdot \frac{4-2t}{-t^2+4t+4} \cdot \frac{-2t^2+8t+8}{(4-2t)^2} dt$$

$$= 2 \int_0^{2\sqrt{2}-2} \frac{1}{t^2+4} dt = \left[\tan^{-1}\left(\frac{t}{2}\right) \right]_0^{2\sqrt{2}-2} = \tan^{-1}(\sqrt{2}-1) = \frac{\pi}{8} \blacksquare$$

92. 제 2종 오일러 치환 : $\sqrt{-x^2+x+2} = xt + \sqrt{2}$, $x = \frac{1-2\sqrt{2}t}{t^2+1}$, $dx = \frac{2\sqrt{2}t^2-2t-2\sqrt{2}}{(t^2+1)^2} dt$

$$\int_1^{\frac{25}{73}} \frac{1}{x\sqrt{-x^2+x+2}} dx$$

$$= \int_0^{\frac{4\sqrt{2}}{25}} \frac{t^2+1}{1-2\sqrt{2}t} \cdot \frac{t^2+1}{-\sqrt{2}t^2+t+\sqrt{2}} \cdot \frac{2\sqrt{2}t^2-2t-2\sqrt{2}}{(t^2+1)^2} dt$$

$$= \int_0^{\frac{4\sqrt{2}}{25}} \frac{-2}{-2\sqrt{2}t+1} dt = \frac{1}{\sqrt{2}} \int_0^{\frac{4\sqrt{2}}{25}} \frac{-2\sqrt{2}}{-2\sqrt{2}t+1} dt = \left[\frac{1}{\sqrt{2}} \ln |2\sqrt{2}t-1| \right]_0^{\frac{4\sqrt{2}}{25}}$$

$$= \sqrt{2} \ln \frac{3}{5} \blacksquare$$

93. 제 3종 오일러 치환 : $\sqrt{-(x-2)(x-1)} = (x-2)t$, $x = \frac{-2t^2-1}{-t^2-1}$, $dx = \frac{2t}{(-t^2-1)^2} dt$

$$\int_1^2 \frac{x^2}{\sqrt{-x^2+3x-2}} dx = \int_0^{-\infty} \left(\frac{-2t^2-1}{-t^2-1} \right)^2 \cdot \frac{-t^2-1}{t} \cdot \frac{2t}{(-t^2-1)^2} dt$$

$$\begin{aligned}
&= -2 \int_0^{-\infty} \frac{(2t^2+1)^2}{(t^2+1)^3} dt = -2 \int_0^{-\infty} \left(\frac{4}{t^2+1} - \frac{4}{(t^2+1)^2} + \frac{1}{(t^2+1)^3} \right) dt \\
&= [-8 \tan^{-1} t + 8I(t) - 2J(t)]_0^{-\infty} = \left[-\frac{19}{4} \tan^{-1} t + \frac{t(13t^2+11)}{4(t^2+1)^2} \right]_0^{-\infty} = \frac{19}{8} \pi \blacksquare
\end{aligned}$$

$$\begin{aligned}
& \ast \quad t = \tan u, \quad dt = \sec^2 u du, \quad I(t) = \int_0^{-\infty} \frac{1}{(t^2+1)^2} dt = \int_0^{-\frac{\pi}{2}} \cos^2 u du \\
&= \frac{1}{2} \int_0^{-\frac{\pi}{2}} (1 + \cos 2u) du = \left[\frac{1}{2} u + \frac{1}{4} \sin 2u \right]_0^{-\frac{\pi}{2}} = \left[\frac{1}{2} \tan^{-1} t + \frac{t}{2(1+t^2)} \right]_0^{-\infty}
\end{aligned}$$

$$\begin{aligned}
& \ast \ast \quad t = \tan u, \quad dt = \sec^2 u du, \quad J(t) = \int_0^{-\infty} \frac{1}{(t^2+1)^3} dt = \int_0^{-\frac{\pi}{2}} \cos^4 u du \\
&= \frac{1}{4} \int_0^{-\frac{\pi}{2}} (1 + \cos 2u)^2 du = \frac{1}{4} \int_0^{-\frac{\pi}{2}} (1 + 2\cos 2u + \cos^2 2u) du \\
&= \frac{1}{4} \int_0^{-\frac{\pi}{2}} \left(1 + 2\cos 2u + \frac{1 + \cos 4u}{2} \right) du = \frac{1}{8} \int_0^{-\frac{\pi}{2}} (3 + 4\cos 2u + \cos 4u) du \\
&= \left[\frac{3}{8} u + \frac{1}{4} \sin 2u + \frac{1}{32} \sin 4u \right]_0^{-\frac{\pi}{2}} = \left[\frac{3}{8} \tan^{-1} t + \frac{t(3t^2+5)}{8(t^2+1)^2} \right]_0^{-\infty}
\end{aligned}$$

94. $t = \cos \theta, \quad dt = -\sin \theta d\theta, \quad u = \frac{t-1}{t+1}, \quad t = \frac{1+u}{1-u}, \quad dt = \frac{2}{(1-u)^2} du$

$$\begin{aligned}
& \int_0^{\frac{\pi}{4}} \frac{\sin^3(\theta/2)}{\cos(\theta/2) \cdot \sqrt{\cos^3 \theta + \cos^2 \theta + \cos \theta}} d\theta \\
&= \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{(1 - \cos \theta) \sin \theta}{(1 + \cos \theta) \sqrt{\cos^3 \theta + \cos^2 \theta + \cos \theta}} d\theta = \frac{1}{2} \int_1^{\frac{1}{\sqrt{2}}} \frac{t-1}{(t+1) \sqrt{t^3+t^2+t}} dt \\
&= \frac{1}{2} \int_0^{2\sqrt{2}-3} u \cdot \frac{\sqrt{(1-u)^3}}{\sqrt{(1+u)^3 + (1+u)^2(1-u) + (1+u)(1-u)^2}} \cdot \frac{2}{(1-u)^2} du \\
&= \int_0^{2\sqrt{2}-3} \frac{u}{\sqrt{3-2u^2-u^4}} du = \frac{1}{2} \int_0^{2\sqrt{2}-3} \frac{2u}{\sqrt{4-(u^2+1)^2}} du
\end{aligned}$$

$$= \left[\frac{1}{2} \sin^{-1} \left(\frac{u^2 + 1}{2} \right) \right]_0^{2\sqrt{2}-3} = \frac{1}{2} \sin^{-1}(9 - 6\sqrt{2}) - \frac{\pi}{12} \blacksquare$$

$$\begin{aligned} 95. \int_0^{\frac{\pi}{6}} \frac{\tan^4 \theta}{1 - \tan^2 \theta} d\theta &= \int_0^{\frac{\pi}{6}} \frac{\tan^4 \theta - 1}{1 - \tan^2 \theta} d\theta + \int_0^{\frac{\pi}{6}} \frac{1}{1 - \tan^2 \theta} d\theta \\ &= - \int_0^{\frac{\pi}{6}} (\tan^2 \theta + 1) d\theta + \int_0^{\frac{\pi}{6}} \frac{\cos^2 \theta}{\cos^2 \theta - \sin^2 \theta} d\theta = - \int_0^{\frac{\pi}{6}} \sec^2 \theta d\theta + \int_0^{\frac{\pi}{6}} \frac{1 + \cos 2\theta}{2 \cos 2\theta} d\theta \\ &= [-\tan \theta]_0^{\frac{\pi}{6}} + \int_0^{\frac{\pi}{6}} \left(\frac{1}{2} \sec 2\theta + \frac{1}{2} \right) d\theta = \left[-\tan \theta + \frac{1}{4} \ln |\sec 2\theta + \tan 2\theta| + \frac{1}{2} \theta \right]_0^{\frac{\pi}{6}} \\ &= -\frac{1}{\sqrt{3}} + \frac{\pi}{12} + \frac{1}{4} \ln(2 + \sqrt{3}) \blacksquare \end{aligned}$$

$$\begin{aligned} 96. \int_{-\infty}^{\infty} \frac{x^2}{x^4 + 1} dx &= \int_{-\infty}^{\infty} \frac{1}{\left(x - \frac{1}{x}\right)^2 + 2} dx = \int_{-\infty}^{\infty} \frac{1}{x^2 + 2} dx \quad (\because \text{Glasser's Master Theorem}) \\ &= \left[\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x}{\sqrt{2}} \right) \right]_{-\infty}^{\infty} = \frac{\pi}{\sqrt{2}} \blacksquare \end{aligned}$$

$$\begin{aligned} 97. \int_0^{\infty} \exp\left(-x^2 - \frac{1}{x^2}\right) dx &= \int_0^{\infty} \exp\left[-\left(x - \frac{1}{x}\right)^2 - 2\right] dx = \frac{1}{2e^2} \int_{-\infty}^{\infty} \exp\left[-\left(x - \frac{1}{x}\right)^2\right] dx \\ &= \frac{1}{2e^2} \int_{-\infty}^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2e^2} \blacksquare \quad (\because \text{Glasser's Master Theorem}) \end{aligned}$$

$$\begin{aligned} 98. \int_0^1 \frac{1 - x^{99}}{(1+x)(1+x^{100})} dx &= \int_0^1 \frac{(1+x^{100}) - x^{99}(1+x)}{(1+x)(1+x^{100})} dx = \int_0^1 \left(\frac{1}{1+x} - \frac{x^{99}}{1+x^{100}} \right) dx \\ &= \left[\ln|1+x| - \frac{1}{100} \ln|1+x^{100}| \right]_0^1 = \frac{99}{100} \ln 2 \blacksquare \end{aligned}$$

$$\begin{aligned} 99. \int_{-\infty}^{\infty} \exp\left(-\frac{(x^2 - 13x - 1)^2}{611x^2}\right) dx &= \int_{-\infty}^{\infty} \exp\left(-\frac{(x - x^{-1} - 13)^2}{611}\right) dx \\ &= \int_{-\infty}^{\infty} \exp\left(-\frac{(x-13)^2}{611}\right) dx \quad (\because \text{Glasser's Master Theorem}) \end{aligned}$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{611}\right) dx \quad (x \mapsto x+13, \quad dx \mapsto dx) \\
&= \int_{-\infty}^{\infty} \exp(-u^2) \cdot \sqrt{611} du \quad (u = \frac{x}{\sqrt{611}}, \quad dx = \sqrt{611} du) \\
&= \sqrt{611\pi} \blacksquare
\end{aligned}$$

100. $t = \tan x, \quad dt = \sec^2 x dx$

$$\int_0^{\frac{\pi}{2}} \sqrt[n]{\tan x} dx = \int_0^{\infty} \frac{t^{\frac{1}{n}}}{1+t^2} dt, \quad \int_0^{\infty} \frac{x^a}{1+x^2} dx = \frac{\pi}{2} \sec\left(\frac{a\pi}{2}\right) \text{임을 이용하자.}$$

sol 1) $t = e^x, \quad dt = e^x dx$

$$\int_0^{\infty} \frac{x^a}{1+x^2} dx = \int_{-\infty}^{\infty} \frac{e^{(a+1)t}}{1+e^{2t}} dt = \int_{-\infty}^0 \frac{e^{(a+1)t}}{1+e^{2t}} dt + \int_0^{\infty} \frac{e^{(a-1)t}}{1+e^{-2t}} dt$$

한편 $\frac{e^{(a+1)t}}{1-(-e^{2t})}, \frac{e^{(a-1)t}}{1-(-e^{-2t})}$ 는 공비가 각각 $(-e^{2t}), (-e^{-2t})$ 이고 초항이 각각 $e^{(a+1)t}, e^{(a-1)t}$ 인 등비급수의 합이므로 이를 역으로 전개할 수 있다.

$$\begin{aligned}
&\int_{-\infty}^0 \frac{e^{(a+1)t}}{1+e^{2t}} dt + \int_0^{\infty} \frac{e^{(a-1)t}}{1+e^{-2t}} dt \\
&= \int_{-\infty}^0 \sum_{n=0}^{\infty} e^{(a+1)t} \cdot (-e^{2t})^n dt + \int_0^{\infty} \sum_{n=0}^{\infty} e^{(a-1)t} \cdot (-e^{-2t})^n dt \\
&= \sum_{n=0}^{\infty} (-1)^n \left(\int_{-\infty}^0 e^{(2n+1+a)t} dt + \int_0^{\infty} e^{-(2n+1-a)t} dt \right) \\
&= \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2n+1+a} + \frac{1}{2n+1-a} \right) = 2 \sum_{n=0}^{\infty} (-1)^n \left(\frac{2n+1}{(2n+1)^2 - a^2} \right) \cdots \quad [\#]
\end{aligned}$$

한편 $\sec(z)$ 의 Mittag-Leffler 전개식은

$$\sec(z) = \pi \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{(\pi/2)^2 (2n+1)^2 - z^2}$$

이므로 (참고 : <https://cdlibb.github.io/WandW/CMA07-3-FactorTheoremMN.html#7.4theexpansionofaclassoffunctionsinrationalfractions.>)

$z = \frac{a\pi}{2}$ 를 대입하고 $\frac{\pi}{2}$ 를 곱하면

$$\frac{\pi}{2} \sec\left(\frac{a\pi}{2}\right) = \frac{\pi^2}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{(\pi/2)^2 (2n+1)^2 - (a\pi/2)^2} = 2 \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{(2n+1)^2 - a^2}$$

을 얻고 이는 [#]의 전개식과 일치한다.

$$\therefore \int_0^{\infty} \frac{x^a}{1+x^2} dx = \frac{\pi}{2} \sec\left(\frac{a\pi}{2}\right) \blacksquare$$

sol 2) $t = x^2$, $dt = 2x dx$

$$\begin{aligned} \int_0^{\infty} \frac{x^a}{1+x^2} dx &= \frac{1}{2} \int_0^{\infty} \frac{t^{(a-1)/2}}{1+t} dt = \frac{1}{2} B\left(\frac{a+1}{2}, 1 - \frac{a+1}{2}\right) \\ &= \frac{1}{2} \Gamma\left(\frac{a+1}{2}\right) \Gamma\left(1 - \frac{a+1}{2}\right) = \frac{\pi}{2} \csc\left(\frac{(a+1)\pi}{2}\right) = \frac{\pi}{2} \sec\left(\frac{a\pi}{2}\right) \blacksquare \end{aligned}$$

※ 베타함수 $B(a, b)$ 는 다음과 같이 정의되는 이변수 함수이다.

$$B(a, b) := \int_0^{\infty} \frac{z^{a-1}}{1+z^{a+b}} dz$$

또한 이는 다음과 같이 감마함수의 비로 표현할 수 있다.

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

감마함수 $\Gamma(z)$ 는 다음과 같이 정의되며, 이는 계승 함수($n!$)의 정의역을 실수부가 양수인 복소수 전체로 확장(음의 정수는 제외)한 것이다. (29번 문제의 1번 솔루션은 이 감마함수의 정의를 이용한 것으로, 자연수 n 에 대하여 $\Gamma(n) = (n-1)!$ 이다.) 감마함수는 계승함수를 해석적으로 확장한 것이므로 팩토리얼의 성질이 그대로 성립하여, $\Gamma(x+1) = x\Gamma(x)$ 이고 $\Gamma(1) = 1$ 이다.)

$$\Gamma(z) := \int_0^{\infty} t^{z-1} e^{-t} dt \quad (Re(z) > 0)$$

감마함수에 대하여 다음과 같은 오일러 반사 공식이 성립하며, 상술한 2번 솔루션의 마지막

막 단계에서 이 공식이 사용되었다.

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$$

위 공식은 Residue Theorem을 사용하여 직접 복소적분을 하거나 감마함수의 정의식을 대입하여 증명할 수 있으며, 본문에서는 복소적분을 사용하지 않는다는 철학을 유지하기 위해 다른 증명법을 제시한다. 복소적분을 포함한 다른 증명법에 관심이 있는 독자는 아래 링크의 증명들을 참고하기 바란다.

<https://math.stackexchange.com/questions/714482/prove-that-gamma-p-times-gamma-1-p-frac-pi-sin-p-pi-forall-p-in-noredirect=1&lq=1>

pf) 감마함수의 바이어슈트라스 무한곱 꼴을 생각하자. (감마함수는 실제로 여러 가지 형태로 정의되고 이들은 모두 동치이다. 상술한 정의는 적분 형태이다.)

$$\frac{1}{\Gamma(p)} = pe^{\gamma p} \prod_{n=1}^{\infty} \left(1 + \frac{p}{n}\right) e^{-\frac{p}{n}}$$

$$\frac{1}{\Gamma(p)} \cdot \frac{1}{\Gamma(-p)} = pe^{\gamma p} \prod_{n=1}^{\infty} \left(1 + \frac{p}{n}\right) e^{-\frac{p}{n}} \cdot (-p)e^{-\gamma p} \prod_{n=1}^{\infty} \left(1 - \frac{p}{n}\right) e^{\frac{p}{n}} = -p^2 \prod_{n=1}^{\infty} \left(1 - \frac{p^2}{n^2}\right)$$

한편 감마함수의 성질에 의해 $\Gamma(1-p) = -p\Gamma(-p)$ 이므로

$$\frac{1}{\Gamma(1-p)\Gamma(p)} = p \prod_{n=1}^{\infty} \left(1 - \frac{p^2}{n^2}\right)$$

이고 바이어슈트라스 분해 정리에 의해

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2 \pi^2}\right)$$

임을 이용하면

$$\frac{1}{\Gamma(1-p)\Gamma(p)} = p \prod_{n=1}^{\infty} \left(1 - \frac{p^2}{n^2}\right) = \frac{\sin(\pi p)}{\pi}$$

연고 증명이 완료되었다. ■

$$\therefore \int_0^{\frac{\pi}{2}} \sqrt[n]{\tan x} dx = \int_0^{\infty} \frac{t^{\frac{1}{n}}}{1+t^2} dt = \frac{\pi}{2} \sec\left(\frac{\pi}{2n}\right) \blacksquare \quad (2 \leq n \in \mathbb{N})$$